TABELA OSNOVNIH INTEGRALA

1. $\int x^p\,dx = \frac{x^{p+1}}{p+1} + C \qquad (p \in \mathbb{R} \wedge p \neq -1)$
2. $\int e^x\,dx = e^x + C$
3. $\int \frac{dx}{x} = \ln|x| + C$
4. $\int \sin x\,dx = -\cos x + C$
5. $\int \cos x\,dx = \sin x + C$
6. $\int \frac{dx}{\cos^2 x} = tg\,x + C$
7. $\int \frac{dx}{\sin^2 x} = -ctg\,x + C$
8. $\int \frac{dx}{\sqrt{1-x^2}} = \begin{cases} \arcsin x + C \\ -\arcsin x + C \end{cases}$
9. $\int \frac{dx}{1+x^2} = \begin{cases} arctg\,x + C \\ -arctg\,x + C \end{cases}$
10. $\int a^x\,dx = \frac{a^x}{\ln a} + C \qquad (a > 0)$
11. $\int \frac{dx}{\sqrt{x^2 \pm 1}} = \ln\left|x + \sqrt{x^2 \pm 1}\right| + C$
12. $\int dx = x + C$
13. $\int sh\,x\,dx = ch\,x + C$
14. $\int ch\,x\,dx = sh\,x + C$
15. $\int \frac{dx}{sh^2 x} = cth\,x + C$
16. $\int \frac{dx}{ch^2 x} = tg\,x + C$
17. $\int \frac{dx}{x+\alpha} = \ln|x+\alpha| + C$

*- zadaci s ispita!

27.02.2010.

NEODREĐENI INTEGRALI

Utorak

$F'(x) = f(x) \Rightarrow \int f(x)\,dx = F(x) + C \qquad C \in \mathbb{R} \qquad C - \text{const.}$

$(\sin x)' = \cos x \Rightarrow \int \cos x\,dx = \sin x + C$

Osobine neodređenog integrala

a) $\int 0\,dx = C$

b) $\int c\,f(x)\,dx = c \cdot \int f(x)\,dx$

c) $\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int \left(\sum_{k=1}^{n} f_k(x) \right) dx = \sum_{k=1}^{n} \int f_k(x)\,dx$

d) $\int f'(x)\,dx = f(x) + C$

e) $\left[\int f(x)\,dx\right]' = f(x)$

PRIMJERI:

① $\int (5x^3 - 4x^2 + 7x - 6)\,dx =$

$= \int 5x^3\,dx - \int 4x^2\,dx + \int 7x\,dx - \int 6\,dx =$

$= 5\int x^3\,dx - 4\int x^2\,dx + 7\int x\,dx - 6\int dx =$

$= 5 \cdot \frac{x^4}{4} - 4\,\frac{x^3}{3} + 7 \cdot \frac{x^2}{2} - 6x + C$

② $\int \frac{x^2 + 4x - 12}{x^2}\,dx = \int \left(\frac{x^2}{x^2} + \frac{4x}{x^2} - \frac{12}{x^2} \right) dx =$

$= \int \frac{x^2}{x^2}\,dx + \int \frac{4x}{x^2}\,dx - \int \frac{12}{x^2}\,dx = \int dx + 4 \cdot \int \frac{1}{x}\,dx - 12 \int x^{-2}\,dx$

$= x + 4\ln|x| - 12 \cdot \frac{x^{-1}}{-1} + C = x + 4\ln|x| + 12\,\frac{1}{x} + C$

$= x + 4\ln|x| + \frac{12}{x} + C$

③ $\int \sqrt[3]{x}\,dx = \int x^{\frac{1}{3}}\,dx = \frac{x^{\frac{1}{3}+1}}{\frac{1}{3}+1} + C = \frac{x^{\frac{4}{3}}}{\frac{4}{3}} + C = \frac{3}{4}\sqrt[3]{x^4} + C$

④ $\int \left(1 - \frac{1}{x^2}\right) \cdot \sqrt{x\sqrt{x}}\,dx = \int \left(1 - \frac{1}{x^2}\right) \cdot \sqrt{\sqrt{x^3}} = \int \left(1 - \frac{1}{x^2}\right) x^{\frac{3}{4}}\,dx$

$= \int x^{\frac{3}{4}}\,dx - \int \frac{1}{x^2} \cdot x^{\frac{3}{4}}\,dx = \frac{x^{\frac{7}{4}}}{\frac{7}{4}} - \int x^{-2} \cdot x^{\frac{3}{4}}\,dx =$

$= \frac{4}{7}\sqrt[4]{x^7} - \int x^{-2+\frac{3}{4}}\, dx = \frac{4}{7}\sqrt[4]{x^7} - \int x^{-\frac{5}{4}}\, dx =$

$= \frac{4}{7}\sqrt[4]{x^7} - \frac{x^{-\frac{1}{4}}}{-\frac{1}{4}} + C = \frac{4}{7}\sqrt[4]{x^7} + 4\cdot\frac{1}{\sqrt[4]{x}} + C = \frac{4}{7}\sqrt[4]{x^7} + \frac{4}{\sqrt[4]{x}} + C$

(5) $\int \frac{6x^3 - x^2\sqrt[3]{x^2} + \sqrt{\frac{1}{x}}}{\sqrt[6]{x}}\, dx =$

$= \int \frac{6x^3}{\sqrt[6]{x}}\, dx - \int \frac{x^2\sqrt[3]{x^2}}{\sqrt[6]{x}} + \int \frac{\sqrt{\frac{1}{x}}}{\sqrt[6]{x}} =$

$= 6\int \frac{x^3}{x^{\frac{1}{6}}}\, dx - \int \frac{x^2 x^{\frac{2}{3}}}{x^{\frac{1}{6}}} + \int \frac{x^{-\frac{1}{2}}}{x^{\frac{1}{6}}} = 6\int x^{\frac{17}{6}}\, dx - \int x^{\frac{15}{6}}\, dx + \int x^{-\frac{4}{6}}\, dx =$

$= 6\cdot\frac{x^{\frac{23}{6}}}{\frac{23}{6}} - \frac{x^{\frac{21}{6}}}{\frac{21}{6}} + \frac{x^{\frac{2}{6}}}{\frac{2}{6}} + C = \frac{6}{23}\sqrt[6]{x^{23}} - \frac{2}{7}\sqrt{x^7} + 3\cdot\sqrt[3]{x} + C$

(6) $\int tg^2 x\, dx = \int \frac{\sin^2 x}{\cos^2 x}\, dx = \int \frac{1-\cos^2 x}{\cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx - \int \frac{\cos^2 x}{\cos^2 x}\, dx$

$= tg\, x - x + C$

(7) $\int \frac{1}{\sin^2 x \cos^2 x}\, dx = \int \frac{\sin^2 x + \cos^2 x}{\sin^2 x \cos^2 x}\, dx =$

$= \int \frac{\sin^2 x}{\sin^2 x \cos^2 x}\, dx + \int \frac{\cos^2 x}{\sin^2 x \cos^2 x}\, dx = \int \frac{1}{\cos^2 x}\, dx + \int \frac{1}{\sin^2 x}\, dx =$

$= tg\, x - ctg\, x + C$

(8) $\int \frac{x^2}{x+1}\, dx = \int \frac{x^2-1+1}{x+1}\, dx = \int \frac{(x^2-1)}{x+1}\, dx + \int \frac{1}{x+1}\, dx =$

$= \int (x-1)\, dx + \int \frac{1}{x+1}\, dx = \int x\, dx - \int dx + \int \frac{1}{x+1}\, dx =$

$= \frac{x^2}{2} - x + \ln|x+1| + C$

⑨ $\int \frac{x^2}{x-2}\,dx = \int \frac{x^2+4-4}{x-2}\,dx = \int \frac{x^2-4}{x-2}\,dx + \int \frac{4}{x-2}\,dx =$

$= \int (x+2)\,dx + \int \frac{4}{x-2}\,dx = \int (x+2)\,dx + 4 \int \frac{dx}{x-2} =$

$= \int x\,dx + 2\int dx + 4 \int \frac{dx}{x-2} = \frac{x^2}{2} + 2x + 4\ln|x-2| + C$

ili:

$x^2 : (x-2) = x + 2 + \frac{4}{x-2}$

$\frac{\pm x^2 \mp 2x}{\pm 2x \mp 4}$

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$\int \frac{x^2}{x-2}\,dx = \int \left(x + 2 + \frac{4}{x-2}\right) dx = \ldots$

⑩ $\int \frac{x^4}{x-3}\,dx =$

$x^4 : (x-3) = x^3 + 3x^2 + 9x + 27 + \frac{81}{x-3}$

$\vdots$

$= \int x^3\,dx + 3\cdot\int x^2\,dx + 9\cdot\int x\,dx + 27\int dx + 81\cdot\int \frac{dx}{x-3} =$

$= \frac{x^4}{4} + \frac{x^3}{3} + \frac{9x^2}{2} + 27x + 81\ln|x-3| + C$

Za vježbu:

ⓐ $\int (\sqrt{x}+3)^2\,dx$

ⓑ $\int \frac{x^3 - x\sqrt{x}}{\sqrt[4]{x}}$

ⓒ $\int \frac{\cos 2x}{\cos x + \sin x}\,dx$

ⓓ $\int \operatorname{ctg}^2 x\,dx$

ⓔ $\int \sqrt{x\cdot\sqrt{x\sqrt{x}}}\,dx$

ⓕ $\int \frac{\sqrt{1-x^2} - x^2 + x^4}{1-x^2}\,dx$

Metoda zamjene promjenljive:

I tip:

$$\int f(ax+b)\,dx \qquad \begin{array}{l} ax+b=t \\ a\,dx = dt \end{array}$$

1. $\int (3x+5)^2\,dx = \left| \begin{array}{l} 3x+5=t \\ 3dx=dt \\ dx=\frac{1}{3}dt \end{array} \right| = \int t^2 \cdot \frac{1}{3}dt = \frac{1}{3}\int t^2\,dt =$

$= \frac{1}{3}\cdot\frac{t^3}{3} + C = \frac{(3x+5)^3}{9} + C$

2. $\int \cos(2-6x)\,dx = \left| \begin{array}{l} 2-6x=t \\ -6dx=dt \\ dx=-\frac{1}{6}dt \end{array} \right| = \int \cos t \cdot \left(-\frac{1}{6}\right) dt = -\frac{1}{6}\int \cos t\,dt =$

$= -\frac{1}{6}\sin t + C = -\frac{1}{6}\sin(2-6x) + C$

3. $\int e^{4x+7}\,dx = \frac{1}{4}e^{4x+7} + C$

Za vježbu:

a) $\int \frac{1}{\cos^2(4x-1)}\,dx$

b) $\int e^{2x+2}\,dx$

c) $\int (6x-5)^4\,dx$

II tip:

$$\int \frac{dx}{ax^2+b}, \quad \int \frac{dx}{\sqrt{ax^2+b}}, \quad \text{smjena: } \sqrt{|a|}\cdot x = \sqrt{|b|}\cdot t$$

$$\int \frac{dx}{x^2-1} = \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$$

① $$\int \frac{dx}{\underbrace{4x^2}_{(2x)^2}-\underbrace{9}_{3^2}} = \left|\begin{array}{l} 2x=3t \\ 2dx=3dt \\ dx=\frac{3}{2}dt \end{array}\right| = \int \frac{\frac{3}{2}dt}{(3t)^2-9} = \frac{3}{2}\int\frac{dt}{9t^2-9} = \frac{3}{2}\int\frac{dt}{9(t^2-1)}$$

$$= \frac{3}{18}\int\frac{dt}{x^2-1} = \frac{1}{6}\cdot\frac{1}{2}\cdot\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{12}\ln\left|\frac{t-1}{t+1}\right| + C =$$

greška!

$$= \frac{1}{12}\ln\left|\frac{2x-3}{2x+3}\right| + C$$

② $$\int\frac{dx}{\sqrt{3x^2+16}} = \left|\begin{array}{l} \sqrt{3}\,x = 4t \\ \sqrt{3}\,dx = 4dt \\ dx = \frac{4dt}{\sqrt{3}} \end{array}\right| = \int\frac{\frac{4dt}{\sqrt{3}}}{\sqrt{16t^2+16}} = \frac{4}{\sqrt{3}}\int\frac{dt}{\sqrt{16(t^2+1)}} =$$

$$= \frac{1}{\sqrt{3}}\int\frac{dt}{\sqrt{t^2+1}} = \frac{1}{\sqrt{3}}\ln\left|t+\sqrt{t^2+1}\right| + C = \frac{1}{\sqrt{3}}\ln\left|\frac{\sqrt{3}x}{4}+\sqrt{\frac{3x^2}{16}+1}\right| + C$$

Za vježbu:

a) $\int \frac{dx}{2x^2+49}$

b) $\int \frac{dx}{9x^2-25}$

c) $\int \frac{dx}{\sqrt{4x^2-1}}$

d) $\int \frac{dx}{\sqrt{3-9x^2}}$

III tip:

$$\int \frac{dx}{ax^2+bx+c}, \quad \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

Kanonski oblik:

$$ax^2+bx+c = a(x+\alpha)^2+\beta$$

① $I = \int \frac{dx}{x^2+6x+13} = \left| \begin{array}{l} x^2+6x+13= \\ x^2+2\cdot x\cdot 3+3^2+4= \\ =(x+3)^2+4 \end{array} \right| = \int \frac{dx}{(x+3)^2+4} \left| \begin{array}{l} x+3=2t \\ dx=2dt \end{array} \right| =$

$= \int \frac{2\,dt}{4t^2+4} = \frac{2}{4}\int \frac{dt}{t^2+1} = \frac{1}{2}\operatorname{arctg} t = \frac{1}{2}\operatorname{arctg}\frac{x+3}{2} + C$

② $I = \int \frac{dx}{3x^2-2x-1} = \frac{1}{3}\int \frac{dx}{\left(x-\frac{1}{3}\right)^2-\frac{4}{9}} = \frac{1}{3}\int \frac{\frac{2}{3}dt}{\frac{4}{9}t^2-\frac{4}{9}} =$

$= \frac{1}{3}\cdot\frac{\frac{2}{3}}{\frac{4}{9}}\int \frac{dt}{t^2-1} = \frac{1}{2}\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C = \frac{1}{4}\cdot\ln\left|\frac{\frac{3x-1}{2}-1}{\frac{3x-1}{2}+1}\right| + C$

$= \frac{1}{4}\ln\left|\frac{3x-3}{3x+1}\right| + C$

smjena:

$x-\frac{1}{3}=\frac{2}{3}t$

$dx=\frac{2}{3}dt$

$3x-1=2t$

$3x^2-2x-1 = 3\left(x^2-\frac{2x}{3}-\frac{1}{3}\right) =$

$=3\left[x^2-2\cdot x\cdot\frac{1}{3}+\left(\frac{1}{3}\right)^2-\left(\frac{1}{3}\right)^2-\frac{1}{3}\right] = 3\cdot\left[\left(x-\frac{1}{3}\right)^2-\frac{1}{9}-\frac{1}{3}\right]$

$=3\cdot\left[\left(x-\frac{1}{3}\right)^2-\frac{4}{9}\right]$

③ $I = \int \frac{dx}{\sqrt{6-x-x^2}} = \int \frac{dx}{\sqrt{\frac{25}{4}-\left(x+\frac{1}{2}\right)^2}} = \left| \begin{array}{ll} x+\frac{1}{2}=\frac{5}{2}t \mid \cdot 2 & 2x+1=5t \\ dx=\frac{5}{2}dt & t=\frac{2x+1}{5} \end{array} \right|$

$= \int \frac{\frac{5}{2}dt}{\sqrt{\frac{25}{4}-\frac{25}{4}t^2}} = \frac{5}{2}\int \frac{dt}{\sqrt{\frac{25}{4}(1-t^2)}} = \cancel{\frac{5}{2}}\cdot\cancel{\frac{2}{5}}\cdot\int\frac{dt}{\sqrt{1-t^2}} =$

$= \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + C = \arcsin\left(\frac{2x+1}{5}\right) + C$

$-x^2-x-6 = -\left(x^2+2\cdot x\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2-6\right] = -\left[\left(x+\frac{1}{2}\right)^2-\frac{1}{4}-6\right] = -\left[\left(x+\frac{1}{2}\right)^2-\frac{25}{4}\right] = \frac{25}{4}-\left(x+\frac{1}{2}\right)^2$

za vježbu:

a) $\int \frac{dx}{2x^2+2x+1}$

c) $\int \frac{dx}{3x^2-10x+3}$

b) $\int \frac{dx}{\sqrt{-2x^2+5x}}$???

d) $\int \frac{dx}{\sqrt{x^2+4x+7}}$

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Utorak

IV tip:

$$\int \frac{f'(x)}{f(x)}\,dx, \quad \int \frac{f'(x)}{\sqrt{f(x)}}\,dx$$

$f(x)=t \Rightarrow f'(x)\,dx=dt$

(1) $\int \frac{f'(x)}{f(x)}\,dx = \left| f(x)=t \right| = \int \frac{dt}{t} = \ln|t|+C$
$= \ln|f(x)|+C$

(2) $\int \frac{f'(x)}{\sqrt{f(x)}} = \left| f(x)=t \right| = \int \frac{dt}{\sqrt{t}} = \int t^{-\frac{1}{2}}\,dt = \frac{t^{\frac{1}{2}}}{\frac{1}{2}} + C = 2\sqrt{t}+C$
$= 2\sqrt{f(x)}+C$

(3) $\int \frac{3x^2-4x+5}{x^3-2x^2+5x+8}\,dx = \ln|x^3-2x^2+5x+8|+C$

(4) $\int \operatorname{tg}x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\int \frac{-\sin x}{\cos x}\,dx = -\ln|\cos x|+C$

(5) $\int \frac{x}{x^2-1}\,dx = \frac{1}{2}\cdot\int \frac{2x\,dx}{x^2-1} = \frac{1}{2}\ln|x^2-1|+C$

(6) $\int \frac{\cos x}{\sqrt{\sin x}}\,dx \overset{\text{zad. tip 2.}}{=} 2\sqrt{\sin x}+C$

(7) $\int \frac{x^3}{\sqrt{x^4+5}}\,dx = \frac{1}{4}\int \frac{4x^3}{\sqrt{x^4+5}}\,dx = \frac{1}{4}\cdot 2\sqrt{x^4+5}+C = \frac{1}{2}\sqrt{x^4+5}+C$

Za vježbu:

a) $\int \operatorname{ctg} x\,dx$

c) $\int \frac{x-3}{x^2-6x+7}\,dx$

b) $\int \frac{x^2}{x^3-3}\,dx$

d) $\int \frac{x+2}{\sqrt{x^2+4x+9}}\,dx$

V tip:

$\int \frac{mx+n}{ax^2+bx+c}\,dx$, $\int \frac{mx+n}{\sqrt{ax^2+bx+c}}\,dx$ $(a\neq 0,\ m\neq 0)$

(1) $I = \int \frac{4x-6}{x^2-6x+8}$

$(x^2-6x+8)' = 2x-6$

$4x-6 = a\cdot(2x-6)+b$

$4x-6 = 2ax-6a+b$

uz x: $2a=4 \Rightarrow a=2$

uz x^0: $-6a+b=-6$ $b=6$

$I = \int \frac{2(2x-6)+6}{x^2-6x+8}\,dx = 2\int \frac{(2x-6)}{x^2-6x+8}\,dx + 6\cdot\underbrace{\int \frac{dx}{x^2-6x+8}}_{I_1}$

$= 2\ln|x^2-6x+8| + 6\,I_1$

$x^2-6x+8 = x^2-2\cdot x\cdot 3+3^2-1 = (x-3)^2-1$

$I_1 = \int \frac{dx}{(x-3)^2-1}\ \left|\begin{array}{l} x-3=t \\ dx=dt\end{array}\right| = \int \frac{dt}{t^2-1} = \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|+C =$

$= \frac{1}{2}\ln\left|\frac{x-4}{x-2}\right|+C$

$I = 2\ln|x^2-6x+8| + 3\ln\left|\frac{x-4}{x-2}\right|+C$

② $I = \int \frac{2x+3}{\sqrt{2x-x^2}}\,dx =$

$(2x-x^2)' = 2-2x = -2x+2$ $\qquad$ uz x: $2x = -2ax \Rightarrow a = -1$

$2x+3 = a\cdot(2-2x)+b$ $\qquad$ sl. član (konst) uz x^0: $2a+b=3 \Rightarrow b=5$

$2x+b = 2a-2ax+b$

$I = \int \frac{-(2-2x)+5}{\sqrt{2x-x^2}} = -\int \frac{2-2x}{\sqrt{2x-x^2}}\,dx + \int \frac{5}{\sqrt{2x-x^2}}\,dx =$

$= -\int \frac{2-2x}{\sqrt{2x-x^2}}\,dx + \underbrace{5\int \frac{dx}{\sqrt{2x-x^2}}}_{I_1} = -2\cdot\sqrt{2x-x^2} + 5\,I_1$

$2x-x^2 = -(x^2-2x) = -(x^2-2\cdot1\cdot x+1-1) = -[(x-1)^2-1] = 1-(x-1)^2$

$I_1 = \int \frac{dx}{\sqrt{1-(1-x)^2}} \left| \begin{matrix} x-1=t \\ dx=dt \end{matrix} \right| = \int \frac{dt}{\sqrt{1-t^2}} = \arcsin t + c = \arcsin(x+1)+c$

$I = -2\cdot\sqrt{2x-x^2} + 5\cdot\arcsin(x+1) + c$

Za vježbu: $\quad$ * - s ispita

a) * $\int \frac{3x+1}{2x^2+5x+4}\,dx$ $\qquad$ c) $\int \frac{x\,dx}{2x^2+3x+5}$

b) * $\int \frac{3x+2}{\sqrt{x^2-8x-9}}\,dx$ $\qquad$ d) $\int \frac{x-2}{\sqrt{x^2+x+3}}\,dx$

VI tip:

$$\int g(f(x))\cdot f'(x)\,dx = \left| \begin{array}{l} f(x)=t \\ f'(x)dx=dt \end{array} \right| = \int g(t)\,dt$$

① $\int \frac{dx}{x\ln^3 x} = \left| \begin{array}{l} \ln x = t \\ \frac{1}{x}dx = dt \end{array} \right| = \int \frac{dt}{t^3} = \int t^{-3}dt = \frac{t^{-2}}{-2} + C =$

$= -\frac{1}{2t^2} + C = -\frac{1}{2\ln^2 x} + C$

② $I = \frac{\operatorname{arctg}^5 x}{1+x^2}\,dx = \left| \begin{array}{l} \operatorname{arctg} x = t \\ \frac{1}{1+x^2}dx = dt \end{array} \right| = \int t^5 dt =$

$= \frac{t^6}{6} + C = \frac{\operatorname{arctg} x^6}{6} + C$

③ $\int x^2 \sqrt[3]{2-x^3}\,dx = \left| \begin{array}{l} 2-x^3 = t^3 \\ -3x^2dx = 3t^2dt \ |:(-3) \\ x^2dx = -t^2dt \end{array} \right| =$

$= -\int \sqrt[3]{t^3}\cdot t^2dt = -\int t^3dt = -\frac{t^4}{4} + C = -\frac{\left(\sqrt[3]{2-x^3}\right)^4}{4} + C$

④ $\int x^3\sqrt{x^2+1}\,dx = \int x^2\cdot x\cdot\sqrt{x^2+1}\,dx = \left| \begin{array}{l} x^2+1 = t^2 \Rightarrow x^2 = t^2-1 \\ 2x\,dx = 2t\,dt \ |:2 \\ x\,dx = t\,dt \end{array} \right| =$

$= \int (t^2-1)\sqrt{t^2}\,t\,dt = \int (t^2-1)\cdot t^2dt = \int (t^4-t^2)\,dt =$

$= \int t^4dt - \int t^2dt = \frac{t^5}{5} - \frac{t^3}{3} + C$

$= \frac{\left(\sqrt{1+x^2}\right)^5}{5} - \frac{\left(\sqrt{1+x^2}\right)^3}{3} + C$

(5) $\int (2x-1)\sqrt[3]{x+1}\,dx = \left| \begin{matrix} x+1 = t^3 \Rightarrow x = t^3-1 \\ dx = 3t^2dt \end{matrix} \right| =$

$= \int (2(t^3-1)-1)\sqrt[3]{t^3}\,3t^2dt = 3\int (2t^3-2-1)\,t^3dt =$

$= 3\int (2t^3-3)\,t^3dt = 3\int 2t^6dt - 3t^3dt =$

$= 3\int 2t^6dt - 3\int 3t^3dt = 6\int t^6dt - 9\int t^3dt =$

$= 6\,\frac{t^7}{7} - 9\,\frac{t^4}{4} + C = \frac{6}{7}\cdot(\sqrt[3]{x+1})^7 - \frac{9}{4}\cdot(\sqrt[3]{x+1})^4 + C$

(6) $\int \frac{dx}{\sqrt{1+e^{2x}}} \overset{:e^x}{\underset{:e^x}{=}} \int \frac{\frac{dx}{e^x}}{\sqrt{\frac{1+e^{2x}}{e^{2x}}}} = \int \frac{e^{-x}dx}{\sqrt{\frac{1}{e^{2x}}+\frac{e^{2x}}{e^{2x}}}} = \int \frac{e^{-x}dx}{\sqrt{\underset{(e^{-x})^2}{e^{-2x}}+1}} =$

$= \left| \begin{matrix} e^{-x} = t \\ -e^{-x}dx = dt \\ e^{-x}dx = -dt \end{matrix} \right| = -\int \frac{dt}{\sqrt{t^2+1}} = -\ln|t+\sqrt{t^2+1}| + C =$

$= -\ln|e^{-x}+\sqrt{e^{-2x}+1}| + C$

(7) $\int \frac{x^2+1}{x^4+1}dx \; \overset{x^2}{\underset{x^2}{}} = \frac{1+\frac{1}{x^2}}{x^2+\frac{1}{x^2}}dx = \left| \begin{matrix} x-\frac{1}{x} = t \\ (1+\frac{1}{x^2})dx = dt \end{matrix} \quad \begin{matrix} x-\frac{1}{x} = t \;|^2 \\ x^2+2\cdot x\cdot\frac{1}{x}+(\frac{1}{x})^2 = t^2 \\ x^2+\frac{1}{x^2} = t^2+2 \end{matrix} \right|$

$= \int \frac{dt}{t^2+2} = \left| t = \sqrt{2}\ldots \right| = \frac{1}{\sqrt{2}}\,\mathrm{arctg}\,\frac{t}{\sqrt{2}} + C$

* (8) $\boxed{I} = \int \frac{e^{3x}(10-2e^{3x})}{2e^{6x}-10e^{3x}+12}dx = \left| \begin{matrix} e^{3x} = t \\ 3e^{3x}dx = dt \;\; |:3 \\ e^{3x}dx = \frac{dt}{3} \end{matrix} \right|$

$= \int \frac{e^{3x}(5-e^{3x})}{e^{6x}-5e^{3x}+6}dx = \frac{1}{3}\int \frac{5-t}{t^2-5t+6}dt$

$(t^2-5t+6)' = 2t-5$ $\qquad -1 = 2a \Rightarrow a = -\frac{1}{2}$

$5-t = a(2t-5)+b$ $\qquad 5 = -5a+b \rightarrow b = \frac{5}{2}$

$5-t = 2at-5a+b$

$$I = \frac{1}{3}\int \frac{-\frac{1}{2}(2t-5)+\frac{5}{2}}{t^2-5t+6}\,dt = \frac{1}{3}\int \frac{-\frac{1}{2}(2t-5)}{t^2-5t+6}\,dt + \frac{1}{3}\int \frac{\frac{5}{2}}{t^2-5t+6}\,dt =$$

$$= -\frac{1}{6}\int \frac{2t-5}{t^2-5t+6}\,dt + \underbrace{\frac{5}{6}\int \frac{dt}{t^2-5t+6}}_{I_1} = -\frac{1}{6}\ln|t^2-5t+6| + \frac{5}{6} I_1$$

$$t^2-5t+6 = t^2 - 2\cdot t\cdot\frac{5}{2} + \left(\frac{5}{2}\right)^2 - \frac{1}{4} = \left(t-\frac{5}{2}\right)^2 - \frac{1}{4}$$

$$I_1 = \int \frac{dt}{\left(t-\frac{5}{2}\right)^2 - \frac{1}{4}} = \left| \begin{matrix} t-\frac{5}{2} = \frac{1}{2}z \quad |\cdot 2 \Rightarrow 2t-5 = z \\ dt = \frac{1}{2}\,dz \end{matrix} \right| =$$

$$\Rightarrow I_1 = \int \frac{\frac{1}{2}\,dz}{\left(\frac{1}{2}z\right)^2 - \frac{1}{4}} = \frac{1}{2}\int \frac{dz}{\frac{1}{4}(z^2-1)} = \frac{\frac{1}{2}}{\frac{1}{4}}\int \frac{dz}{z^2-1} = 2\cdot\frac{1}{2}\ln\left|\frac{z-1}{z+1}\right| + C$$

$$= \ln\left|\frac{2t-6}{2t-4}\right| + C = \ln\left|\frac{t-3}{t-2}\right| + C$$

$$I = -\frac{1}{6}\ln|t^2-5t+6| + \frac{5}{6}\ln\left|\frac{t-3}{t-2}\right| + C$$

$$I = -\frac{1}{6}\ln|e^{6x} - 5\cdot e^{3x} + 6| + \frac{5}{6}\ln\left|\frac{e^{3x}-3}{e^{3x}-2}\right| + C$$

za vježbu:

a) $\int \frac{x^3\,dx}{\sqrt[5]{x^4+2}}$ $\qquad$ d) $\int x(1-x)^{10}\,dx$

b) $\int x^2 e^{x^3}\,dx$ $\qquad$ e) $\int x\sqrt{x-1}\,dx$

c) $\int \frac{x^3\,dx}{x^8-2}$ $\qquad$ f) $\int \frac{x^2+1}{\sqrt{x^6-7x^4+x^2}}\,dx$

g*) $\int \frac{3x+5}{2+\sqrt[3]{x-1}}\,dx$

METODA PARCIJALNE INTEGRACIJE

$$\left.\begin{matrix} u = u(x) \\ v = v(x) \end{matrix}\right\} \Rightarrow \int u\,dv = uv - \int v\,du$$

① $I = \int x e^x\,dx = \left| \begin{matrix} u = x & dv = e^x dx \\ du = dx & v = \int e^x dx = e^x \end{matrix} \right| = xe^x - \int e^x dx = xe^x - e^x + C$

$= e^x(x-1) + C$

pogrešan izbor:

$\left| \begin{matrix} u = e^x & dv = x\,dx \\ du = e^x dx & v = \int x\,dx = \frac{x^2}{2} \end{matrix} \right| \quad I = e^x \cdot \frac{x^2}{2} - \underbrace{\int \frac{x^2}{2} e^x dx}$

komplikovaniji integral od polaznog

② $I = \int x^2 \sin 2x\,dx = \left| \begin{matrix} u = x^2 & dv = \sin 2x\,dx \\ du = 2x\,dx & v = \int \sin 2x\,dx \\ (2x = t) & = -\frac{1}{2}\cos 2x \end{matrix} \right|$

Napomena: $\int \sin \alpha x\,dx = -\frac{1}{\alpha}\cos \alpha x + C \quad \alpha \neq 0$

$\int \cos \alpha x\,dx = \frac{1}{\alpha} \sin \alpha x + C, \quad \alpha \neq 0$

$I = x^2\left(-\frac{1}{2}\cos 2x\right) - \int\left(-\frac{1}{\cancel{2}}\right)\cos 2x \cdot \cancel{2}x\,dx = -\frac{1}{2}x^2\cos 2x + \underbrace{\int x\cos 2x\,dx}_{I_1}$

$I_1 = \int x\cos 2x\,dx = \left| \begin{matrix} u = x & dv = \cos 2x\,dx \\ du = dx & v = \int \cos 2x\,dx = \frac{1}{2}\sin x \end{matrix} \right|$

$= \frac{x}{2}\sin 2x - \int \frac{1}{2}\sin 2x\,dx = \frac{x}{2}\sin 2x - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right) + C$

$I = -\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x + C$

③ $I = \int x^3 \ln x\, dx = \left| \begin{matrix} u = \ln x & dv = x^3 dx \\ du = \frac{1}{x}dx & v = \int x^3 dx = \frac{x^4}{4} \end{matrix} \right|$

$= \ln x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x}\, dx = \frac{x^4}{4}\ln x - \frac{1}{4}\int x^3 dx = \frac{x^4}{4}\ln x - \frac{1}{4}\frac{x^4}{4} + C$

$= \frac{x^4}{4}\ln x - \frac{x^4}{16} + C$

④ $I = \int \arcsin x\, dx = \left| \begin{matrix} u = \arcsin x & dv = dx \\ du = \frac{1}{\sqrt{1-x^2}} & v = x \end{matrix} \right| =$

$= x\arcsin x - \underbrace{\int x \frac{1}{\sqrt{1-x^2}}\, dx}_{I_1}$

$I_1 = \int \frac{x\,dx}{\sqrt{1-x^2}} = \left| \begin{matrix} 1-x^2 = t^2 & x\,dx = -t\,dt \\ -2x\,dx = 2t\,dt & \end{matrix} \right| =$

$= -\int \frac{t\,dt}{\sqrt{t^2}} = -t + C = -\sqrt{1-x^2} + C$

$I = x\arcsin x + \sqrt{1-x^2} + C$

Završiti za zadaću!

⑤* $I = \int \frac{\arcsin \frac{x}{2}}{\sqrt{2-x}}\, dx = \left| \begin{matrix} u = \arcsin\frac{x}{2} & dv = \frac{dx}{\sqrt{2-x}} \\ du = \frac{1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \cdot \frac{1}{2}\, dx & v = \int \frac{dx}{\sqrt{2-x^2}} \end{matrix} \right|$ ...

$\left| \begin{matrix} 2-x = t^2 \\ dx = -2t\,dt \end{matrix} \right| = \ldots$

Završiti za zadaću!

⑥* $I = \int x\sqrt{1-x^2} \arcsin x\, dx = \left| \begin{matrix} u = \arcsin x & dv = x\sqrt{1-x^2}\, dx \\ du = \frac{1}{\sqrt{1-x^2}}\, dx & v = \int x\sqrt{1-x^2} \end{matrix} \right|$

$dx = \left| 1-x^2 = t^2 \ \ldots \right.$

za vježbu:

a) $\int x^2 e^{4x}\,dx$

b) $x^3 \cos x\,dx$

c) $\int \frac{\ln x}{x^6}\,dx$

d*) $\int \frac{\ln^2 x}{x^2}\,dx$

e*) $\int \frac{\ln(x^2+1)}{x^3}\,dx$

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7) $I = \int e^{2x}\cos x\,dx = \left| \begin{matrix} u = e^{2x} & dv = \cos x\,dx \\ du = 2e^{2x}dx & v = \int \cos x\,dx = \sin x \end{matrix} \right| =$

$= e^{2x}\sin x - \int \sin x \, 2e^{2x}\,dx = e^{2x}\sin x - 2\underbrace{\int e^{2x}\sin x\,dx}_{I_1}$

$I_1 = \left| \begin{matrix} u = e^{2x} & dv = \sin x\,dx \\ du = 2e^{2x}dx & v = -\cos x \end{matrix} \right| = -e^{2x}\cos x - \int(-\cos x)\cdot 2e^{2x}\,dx =$

$= -e^{2x}\cos x + 2\underbrace{\int e^{2x}\cos x\,dx}_{I}$

$I = e^{2x}\sin x - 2(-e^{2x}\cos x + 2I)$

$I = e^{2x}\sin x + 2e^{2x}\cos x - 4I$

$5I = e^{2x}(\sin x + 2\cos x) + C \Rightarrow$

$I = \frac{e^{2x}(\sin x + 2\cos x)}{5} + C$

8) $I = \int \sin(\ln x)\,dx \quad \left| \begin{matrix} u = \sin(\ln x) & dv = dx \\ du = \cos(\ln x)\cdot \frac{1}{x}dx & v = x \end{matrix} \right|$

$= \sin(\ln x)\cdot x - \int x\cdot\cos(\ln x)\cdot\frac{1}{x}\,dx \Rightarrow$

$I = \sin(\ln x)\cdot x - \underbrace{\int \cos(\ln x)\,dx}_{I_1}$

$I_1 = \left| \begin{matrix} \cos(\ln x) = u & v = x \\ -\sin(\ln x)\frac{1}{x}dx = du & \end{matrix} \right| = \cos(\ln x)\cdot x + \int \sin(\ln x)\frac{1}{x}\cdot x\,dx$

$= \cos(\ln x)x + I$

$I = \sin(\ln x)x - \cos(\ln x)\cdot x - I$

$I = \frac{x(\sin(\ln x) - \cos(\ln x))}{2} + C$

(9) $I = \int \underbrace{\sqrt{a^2-x^2}}_{u}\,\underbrace{dx}_{dv}$ $\quad \left| \begin{array}{ll} u = \sqrt{a^2-x^2} & dx = dv \\ du = \frac{-2x\,dx}{2\sqrt{a^2-x^2}} & x = v \\ \quad = \frac{-x\,dx}{\sqrt{a^2-x^2}} & \end{array} \right|$

$= x\sqrt{a^2-x^2} - \int x\,\frac{-x\,dx}{\sqrt{a^2-x^2}} = x\sqrt{a^2-x^2} + \underbrace{\int \frac{x^2\,dx}{\sqrt{a^2-x^2}}}_{I_1}$

$I_1 = \int \frac{x^2-a^2+a^2}{\sqrt{a^2-x^2}}\,dx = \int \frac{a^2\,dx}{\sqrt{a^2-x^2}} + \int \frac{x^2-a^2}{\sqrt{a^2-x^2}}\,dx$

$= a^2 \underbrace{\int \frac{dx}{\sqrt{a^2-x^2}}}_{I_2} - \underbrace{\int \frac{a^2-x^2}{\sqrt{a^2-x^2}}\,dx}_{I} =$

$\frac{a^2-x^2}{\sqrt{a^2-x^2}} = \sqrt{\frac{(a^2-x^2)^2}{a^2-x^2}} = \sqrt{a^2-x^2}$

$I_2 = \int \frac{dx}{\sqrt{a^2-x^2}} = \left| \begin{array}{l} x = at \\ dx = a\,dt \end{array} \right| = \int \frac{a\,dt}{\sqrt{a^2-a^2t^2}} = \int \frac{\cancel{a}\,dt}{\sqrt{\cancel{a^2}(1-t^2)}} = \arcsin t + C$

$= \arcsin \frac{x}{a} + C$

$I = x\sqrt{a^2-x^2} + a^2\cdot\arcsin\frac{x}{a} - I$

$2I = x\cdot\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a} + C$

$I = \frac{x\cdot\sqrt{a^2-x^2} + a^2\arcsin\frac{x}{a}}{2} + C$

(10) $\int \frac{x^2\,dx}{(x^2+1)^2} = \left| \begin{array}{ll} u = x & dv = \frac{x\,dx}{(x^2+1)^2} \\ du = dx & v = \int \frac{x\,dx}{(x^2+1)^2} \end{array} \right|$

$x^2+1 = t$

$2x\,dx = dt$

$x\,dx = \frac{dt}{2}$

$v = \int \frac{\frac{dt}{2}}{t^2}$

$v = \frac{1}{2}\int \frac{dt}{t^2}$

$v = \frac{1}{2}\int t^{-2}\,dt$

$v = \frac{-1}{2t} = \frac{-1}{2(x^2+1)}$

$$I = x\cdot\left(-\frac{1}{2(x^2+1)}\right) - \int -\frac{1}{2(x^2+1)}\,dx = -\frac{x}{2(x^2+1)} + \frac{1}{2}\int\frac{dx}{x^2+1} = \frac{-x}{2(x^2+1)} + \frac{1}{2}\operatorname{arctg} x + C$$

Završit za zadaću!

(11) $$\int\frac{dx}{(x^2+a^2)^2} = \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(x^2+a^2)^2}\,dx = \frac{1}{a^2}\left[\underbrace{\int\frac{(x^2+a^2)\,dx}{(x^2+a^2)^2}}_{I_1} - \underbrace{\int\frac{x^2}{(x^2+a^2)^2}\,dx}_{I_2}\right]$$

I_1 - smjena $x = at$

$$I_2 = \int x\cdot\frac{x\,dx}{(x^2+a^2)^2} = \left| \begin{array}{ll} u = x & dv = \frac{x\,dx}{(x^2+a^2)^2} \\ \ldots & \end{array} \right|$$

Završit za zadaću

(12) $$\int\frac{dx}{(x^2+a^2)^3} = \frac{1}{a^2}\int\frac{x^2+a^2-x^2}{(x^2+a^2)^3}\,dx = \ldots$$

Za vježbu:

a) $\int e^{\alpha x}\cos\beta x\,dx$

b) $\int e^{\alpha x}\sin\beta x\,dx$

c) $\int\cos(2\ln x)\,dx$

d) $\int e^{\arccos x}\,dx$

INTEGRACIJA RACIONALNIH FUNKCIJA

$R(x) = \frac{P_m(x)}{Q_n(x)}$ - racionalna funkcija

P_m - polinom stepena m

Q_n - polinom stepena n

$m < n \Rightarrow R(x)$ je prava racionalna funkcija

parcijalni (prosti) razlomci:

$$\frac{\alpha}{(Ax+B)^n},\quad \frac{\alpha x+\beta}{(x^2+px+q)^n} \qquad (\alpha, \beta, A, B, p, q \in \mathbb{R},\ n\in\mathbb{N})$$

① $I = \int \frac{2x-4}{x^2+8x+12}\,dx =$

$x^2+8x+12 = x^2+6x+2x+12 =$
$= x(x+6)+2(x+6) = (x+6)(x+2)$

$\frac{2x-4}{(x+6)(x+2)} = \frac{a}{x+6} + \frac{b}{x+2} \quad / \cdot (x+6)(x+2)$

$2x-4 = a(x+2)+b(x+6)$ $\quad a+b=2$ $\quad a=4$
$2x-4 = ax+2a+bx+6b$ $\quad 2a+6b=-4$ $\quad b=-2$
$2x-4 = x(a+b)+2a+6b$

$I = \int \frac{4}{x+6}dx + \int \frac{-2}{x+2}dx = 4\int\frac{dx}{x+6} - 2\int\frac{dx}{x+2} = 4\ln|x+6| - 2\ln|x+2| + C$

② $I = \int \frac{x\,dx}{(x-2)(x+1)(x+2)}$

$\frac{x}{(x-2)(x+1)(x+2)} = \frac{a}{x-2} + \frac{b}{x+1} + \frac{c}{x+2} \quad /(x-2)(x+1)(x+2)$

$x = a(x+1)(x+2) + b(x-2)(x+2) + c(x-2)(x+1)$
$x=-1 \Rightarrow -1 = b(-3)\cdot 1 \Rightarrow b = \frac{1}{3}$
$x=-2 \Rightarrow -2 = c(-4)(-1) \Rightarrow c = -\frac{1}{2}$
$x=2 \Rightarrow 2 = a\cdot 3\cdot 4 \Rightarrow a = \frac{1}{6}$

$I = \int \frac{\frac{1}{6}}{x-2} + \int \frac{\frac{1}{3}}{x+1} + \int \frac{-\frac{1}{2}}{x+2} = \frac{1}{6}\int\frac{dx}{x-2} + \frac{1}{3}\int\frac{dx}{x+1} - \frac{1}{2}\int\frac{dx}{x+2}$

$= \frac{1}{6}\ln|x-2| + \frac{1}{3}\ln|x+1| - \frac{1}{2}\ln|x+2| + C$

③ $I = \int \frac{dx}{x^3+1}$

$x^3+1 = (x+1)(x^2-x+1)$

$D=-3<0$ (ne može se rastavit na proste faktore)

$$\frac{1}{(x+1)(x^2-x+1)} = \frac{a}{x+1} + \frac{bx+c}{x^2-x+1} \quad \Big| \cdot (x+1)(x^2-x+1)$$

$$1 = a(x^2-x+1) + (bx+c)\cdot(x+1) \qquad a+b=0$$

$$1 = ax^2 - ax + a + bx^2 + bx + cx + c \qquad b+c-a=0$$

$$1 = x^2(a+b) + x(b+c-a) + a+c \qquad a+c=1$$

$$\boxed{\begin{aligned} a &= \tfrac{1}{3} \\ b &= -\tfrac{1}{3} \\ c &= \tfrac{2}{3} \end{aligned}}$$

$$I = \int \left(\frac{\frac{1}{3}}{x+1} + \frac{-\frac{1}{3}x + \frac{2}{3}}{x^2-x+1} \right) dx =$$

$$I = \frac{1}{3}\int \frac{dx}{x+1} - \underbrace{\frac{1}{3}\int \frac{x-2}{x^2-x+1}}_{I_1} = \frac{1}{3}\ln|x+1| - \frac{1}{3} I_1$$

$$(x^2-x+1)' = 2x-1$$

$$x-2 = \alpha(2x-1) + \beta$$

$$x-2 = 2\alpha x - \alpha + \beta$$

$$2\alpha = 1$$

$$-\alpha + \beta = -2$$

$$\boxed{\begin{aligned} \alpha &= \tfrac{1}{2} \\ \beta &= -\tfrac{3}{2} \end{aligned}}$$

$$I_1 = \int \frac{\frac{1}{2}(2x-1) - \frac{3}{2}}{x^2-x+1}\, dx - \underbrace{\frac{3}{2}\int \frac{dx}{x^2-x+1}}_{I_2} = \frac{1}{2}\ln(x^2-x+1) - \frac{3}{2} I_2$$

$$x^2 - x + 1 = x^2 - 2\cdot x \cdot \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \frac{3}{4} = \left(x - \frac{1}{2}\right)^2 + \frac{3}{4}$$

$$I_2 = \int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} = \left| \begin{aligned} x - \tfrac{1}{2} &= \tfrac{\sqrt{3}}{2} t \\ dx &= \tfrac{\sqrt{3}}{2} dt \\ 2x-1 &= \sqrt{3}\, t \end{aligned} \right| = \int \frac{\frac{\sqrt{3}}{2} dt}{\frac{3}{4}t^2 + \frac{3}{4}} = \frac{\frac{\sqrt{3}}{2}}{\frac{3}{4}} \int \frac{dt}{t^2+1} =$$

$$= \frac{4\sqrt{3}}{6} \operatorname{arctg} t + c = \frac{2\sqrt{3}}{3} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + c$$

$$I_1 = \frac{1}{2}\ln(x^2-x+1) - \frac{\cancel{3}}{\cancel{2}} \cdot \frac{\cancel{2}\sqrt{3}}{\cancel{3}} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + c$$

$$I = \frac{1}{3}\ln|x+1| - \frac{1}{6}\ln(x^2-x+1) + \frac{\sqrt{3}}{3} \operatorname{arctg} \frac{2x-1}{\sqrt{3}} + c$$

④ $I = \int \frac{x^2}{x^4+x^2-2}\,dx =$

$x^4+x^2-2 = x^4+2x^2-x^2-2$

$= x^2(x^2-1) + 2(x^2-1) =$

$= (x^2+2)(x^2-1) = (x^2+2)(x-1)(x+1)$

$I = \int \frac{x^2}{x^4+x^2-2}\,dx = \frac{ax+b}{x^2+2} + \frac{c}{x-1} + \frac{d}{x+1} \quad \Big| \; x^4+x^2-2$

$x^2 = (ax+b)(x^2-1) + c(x+1)(x^2+2) + d(x+1)(x^2+2)$

$x=1 \Rightarrow$ $\quad c = \frac{1}{6}$

$x=-1 \Rightarrow$ $\quad d = -\frac{1}{6}$

$x=0 \Rightarrow$ $\quad b = \frac{2}{3}$

$x=2 \Rightarrow \quad 4 = (2a+\frac{2}{3})\cdot 3 + \frac{1}{6}\cdot 8\cdot 3 - \frac{1}{6}\cdot 6 \qquad 6a+2+3-1=4 \qquad \boxed{a=0}$

$I = \int \left(\frac{\frac{2}{3}}{x^2+2} + \frac{\frac{1}{6}}{x-1} - \frac{\frac{1}{6}}{x+1} \right) dx = \frac{2}{3}\int \frac{dx}{x^2+2} + \frac{1}{6}\int \frac{dx}{x-1} - \frac{1}{6}\int \frac{dx}{x+1}$

$I = \frac{2}{3}\,\frac{1}{\sqrt{2}}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{1}{6}\ln|x-1| - \frac{1}{6}\ln|x+1| + C$

$I = \frac{\sqrt{2}}{3}\,\mathrm{arctg}\,\frac{x}{\sqrt{2}} + \frac{1}{6}\ln\left|\frac{x-1}{x+1}\right| + C$

Završit za zadaću!

⑤ $I = \int \frac{1}{(x+3)^3(x-2)}\,dx$

$\frac{1}{(x+3)^3(x-2)} = \frac{a}{x+3} + \frac{bx+c}{(x+3)^2} + \frac{d}{x-2} \quad \Big| \; (x+3)^3(x-2)$

$1 = a(x-2)(x+3)^2 + (bx+c)(x-2)(x+3) + d(x+3)^3$

$x=2 \Rightarrow \quad 1 = d\,(2+3)^3 \qquad 1 = d\cdot 125 \qquad d = \frac{1}{125}$

$x =$

Završit za vježbu!

* ⑥ $\int \frac{2x^3+3x}{x^4+x^2+1}\,dx = \int \frac{x(2x^2+3)}{x^4+x^2+1}\,dx = \left| \begin{array}{l} x^2=t \\ 2x\,dx=dt \\ x\,dx=\frac{dt}{2} \end{array} \right| = \frac{1}{2}\int \frac{2t+3}{t^2+t+1}\,dt \ldots$

I način:

$x^4+x^2+1 = x^4+2x^2-x^2+1 = (x^2+1)^2-x^2 = (x^2+1-x)(x^2+1+x)$

$\frac{2x^3+3x}{(x^2+x+1)(x^2+1-x)} = \frac{ax+b}{x^2-x+1} + \frac{cx+d}{x^2+x+1} \ldots$

⑦* $\int \frac{x^2}{x^3+3x^2+3x+1}\,dx = \int \frac{x^2}{(x+1)^3}\,dx = \left| \begin{array}{l} x+1=t \\ dx=dt \end{array} \right| = \int \frac{(t-1)^2}{t^3}\,dt =$

$= \int \frac{t^2-2t+1}{t^3}\,dt = \int \frac{t^2}{t^3}\,dt + \int \frac{-2t}{t^3}\,dt + \int \frac{dt}{t^3} =$

$= \ln|t| - 2\int \frac{1}{t^2}\,dt + \int t^{-3}\,dt = \ln|t| + \frac{2}{t} + \frac{t^{-2}}{-2} + C$

$= \ln|t| + \frac{2}{t} - \frac{1}{2t^2} + C = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C$

II način

$\frac{x^2}{(x+1)^3} = \frac{a}{x+1} + \frac{b}{(x+1)^2} + \frac{c}{(x+1)^3} \ldots$

⑧ $I = \int \frac{5x^2-12}{(x^2-6x+13)^2}\,dx =$

$x^2-6x+13 = x^2-6x+9+4 = (x-3)^2+4$

$I = \int \frac{5x^2-12}{[(x-3)^2+4]^2} = \left| \begin{array}{l} x-3=2t \\ dx=2dt \\ x=2t+3 \end{array} \right| = \int \frac{5\cdot(2t+3)^2-12}{[(2t)^2+4]^2}\,2dt =$

$I = \int \frac{5(4t^2+12t+9)-12}{[4(t^2+1)]^2}\cdot 2dt = \int \frac{20t^2+60t+45-12}{\cancel{16}_{8}\cdot(t^2+1)^2}\,\cancel{2}dt =$

$$= \frac{1}{8}\left[\underbrace{\int \frac{20t^2}{(t^2+1)^2}\,dt}_{I_1} + \underbrace{\int \frac{60t}{(t^2+1)^2}\,dt}_{I_2} + \underbrace{\int \frac{33}{(t^2+1)^2}\,dt}_{I_3}\right] =$$

$$I_1 = 20 \int \frac{t \cdot t\,dt}{(t^2+1)^2} = \left| \begin{matrix} u = t & dv = \frac{t\,dt}{(t^2+1)^2} \end{matrix} \right.$$

$$I_2 = 60 \int \frac{t}{(t^2+1)^2}\,dt = \left| t^2+1 = z \right| = 60 \cdot \left(-\frac{1}{2}\right) \cdot \frac{1}{t^2+1}$$

$$I_3 = 33 \int \frac{t^2+1-t^2}{(t^2+1)^2}\,dt \quad \ldots$$

završiti za zadaću!

9) $$I = \int \frac{x^3\,dx}{(x^8+1)^2} = \left| \begin{matrix} x^4 = t \\ 4x^3\,dx = dt \\ x^3\,dx = \frac{1}{4}\,dt \end{matrix} \right| =$$

$$= \frac{1}{4} \int \frac{dt}{(t^2+1)^2} \quad \ldots$$

za vježbu:

a) $\int \frac{dx}{x^4+2x^2-3}$

b) $\int \frac{x\,dx}{x^3-3x+2}$

c) $\int \frac{x^2-18x+35}{x^3-2x^2-5x+6}\,dx$

d) $\int \frac{x\,dx}{(x^2+2x+2)^2}$

e) $\int \frac{5x-3}{(x-2)(3x^2+2x-1)}\,dx$

f) $\int \frac{x^2}{(x-1)^2(x^2+1)}\,dx$

16. 03. 2010

(10) $I=\int \frac{x^5+2x^3+4x+4}{x^4+2x^3+2x^2}\,dx$

Funkcija nije prava racionalna pa ćemo polinome podijeliti!

$(x^5+2x^3+4x+4):(x^4+2x^3+2x^2)=x-2+\frac{4x^3+4x^2+4x+4}{x^4+2x^3+2x^2}$ (1)

$\underline{x^5+2x^4+2x^3}$

$=-2x^4+4x+4$

$\underline{-2x^4-4x^3-4x^2}$

$=4x^3+4x^2+4x+4$

$I=\int\left((x-2)+\frac{4x^3+4x^2+4x+4}{x^4+2x^3+2x^2}\right)dx=\int x\,dx-2\int dx+\underbrace{\int\frac{4x^3+4x^2+4x+4}{x^4+2x^3+2x^2}\,dx}_{I_1}$

$=\frac{x^2}{2}-2x+I_1$

$\frac{4x^3+4x^2+4x+4}{x^2(x^2+2x+2)}=\frac{a}{x}+\frac{b}{x^2}+\frac{cx+d}{x^2+2x+2}$

$4x^3+4x^2+4x+4=ax(x^2+2x+2)+b(x^2+2x+2)+(cx+d)x^2$

$4x^3+4x^2+4x+4=ax^3+2ax^2+2ax+bx^2+2bx+2b+cx^3+dx^2$

$a+c=4$ $\boxed{c=4}$

$2a+b+d=4$ $\boxed{d=2}$

$2a+2b=4$ $\boxed{a=0}$

$2b=4 \Rightarrow \boxed{b=2}$

$I_1=\int\left(\frac{2}{x^2}+\frac{4x+2}{x^2+2x+2}\right)dx=2\int\frac{dx}{x^2}+\int\frac{2(2x+2)-2}{x^2+2x+2}\,dx=$

$=-\frac{2}{x}+2\int\frac{2x+2}{x^2+2x+2}\,dx-2\underbrace{\int\frac{dx}{x^2+2x+2}}_{I_2}=$

$=-\frac{2}{x}+2\ln(x^2+2x+2)-2I_2$

$x^2+2x+2=x^2+2x+1+1=(x+1)^2+1$

$I_2=\int\frac{dx}{(x+1)^2+1}=\left|\begin{matrix}x+1=t\\dx=dt\end{matrix}\right|=\int\frac{dt}{t^2+1}=\operatorname{arctg}t+C=\operatorname{arctg}(x+1)+C$

$$y = \frac{x^2}{2} - 2x - \frac{2}{x} + 2\ln(x^2+2x+2) - 2\,\mathrm{arctg}(x+1) + C$$

(11) $$I = \int \frac{x^5+x^4-8}{x^3-4x}\,dx$$

$$(x^5+x^4-8):(x^3-4x) = x^2+x+4 + \frac{4x^2+16x-8}{x^3-4x}$$

$$\begin{array}{l} x^5-4x^3 \\ \hline = x^4+4x^3-8 \\ \quad x^4-4x^2 \\ \hline = 4x^3+4x^2-8 \\ \quad 4x^3-16x \\ \hline = 4x^2+16x-8 \end{array}$$

$$I = \int\left(x^2+x+4+\frac{4x^2+16x-8}{x^3-4x}\right)dx =$$

$$I = \int x^2dx + \int x\,dx + \int 4\,dx + \underbrace{\int \frac{4x^2+16x-8}{x^3-4x}\,dx}_{I_1} =$$

$$I = \frac{x^3}{3} + \frac{x^2}{2} + 4x + I_1$$

$$\frac{4x^2+16x-8}{x\cdot(x^2-4)} = \frac{a}{x} + \frac{b}{x-2} + \frac{c}{x+2} \quad \Big|\cdot x(x^2-4)$$

$$4x^2+16x-8 = a(x^2-4) + b(x+2)\cdot x + c(x-2)\cdot x$$

$$I = \frac{x^3}{3} + \frac{x^2}{2} + 4x + 2\ln|x| + 5\ln|x-2| - 3\ln|x+2| + C$$

Za vježbu:

a) $J = \int \frac{2x^4 - 2x^3 - x^2 + 2}{2x^3 - 4x^2 + 3x - 1} dx$

b) $J = \int \frac{x^4}{(x^2+1)(x+2)} dx$ → izmnožit

c) $J = \int \frac{x^6 - 2x^4 + 3x^3 - 9x^2 + 4}{x^5 - 5x^3 + 4x} dx$

d) $J = \int \frac{x^5}{x^3 + 8} dx$

INTEGRACIJA IRACIONALNIH FUNKCIJA

(funkcije gdje se pojavljuju korijeni)

1° Metoda Ostrogradski

$$\int \frac{P_n(x)}{\sqrt{ax^2+bx+c}} dx = Q_{n-1}(x) \cdot \sqrt{ax^2+bx+c} + \lambda \int \frac{dx}{\sqrt{ax^2+bx+c}}$$

P_n - polinom stepena n

Q_{n-1} - polinom stepena $(n-1)$ s neodređenim koeficijentima

λ - nepoznati koeficijent

① $J = \int \frac{3x^3}{\sqrt{x^2+4x+5}} = (ax^2+bx+c)\sqrt{x^2+4x+5} + \lambda \int \frac{dx}{\sqrt{x^2+4x+5}} \quad \Big/ \frac{d}{dx}$

$$\frac{3x^3}{\sqrt{x^2+4x+5}} = (2ax+b)\sqrt{x^2+4x+5} + (ax^2+bx+c) \cdot \frac{2x+4}{2\sqrt{x^2+4x+5}} + \lambda \frac{1}{\sqrt{x^2+4x+5}}$$

$\Big/ \cdot \sqrt{x^2+4x+5}$

$$3x^3 = (2ax+b)(x^2+4x+5) + (ax^2+bx+c)\cdot(x+2) + \lambda$$

$$3x^3 = 2ax^3 + 8ax^2 + 10ax + bx^2 + 4bx + 5b + ax^3 + 2ax^2 + bx^2 + 2bx + cx + 2c + \lambda$$

$$3x^3 = 3ax^3 + 10ax^2 + 2bx^2 + 10ax + 6bx + cx + 5b + 2c + \lambda$$

$3a = 3 \quad \Rightarrow \boxed{a = 1}$

$10a + 2b = 0 \quad \boxed{b = -5}$

$10a + 6b + c = 0 \quad \boxed{c = 20}$

$5b + 2c + \lambda = 0 \quad \boxed{\lambda = -15}$

$$I = (x^2 - 5x + 20)\sqrt{x^2+4x+5} - 15 \underbrace{\int \frac{dx}{\sqrt{x^2+4x+5}}}_{I_1}$$

$$\int \sqrt{ax^2+bx+c}\, dx = \int \frac{ax^2+bx+c}{\sqrt{ax^2+bx+c}}\, dx$$ – prilagođeno za metodu Ostrogradski

(2) $$\int \sqrt{x^2+1}\, dx = \int \frac{x^2+1}{\sqrt{x^2+1}}\, dx = (ax+b)\cdot\sqrt{x^2+1} + \lambda \int \frac{dx}{\sqrt{x^2+1}} \quad \Big| \frac{d}{dx}$$

$$\frac{x^2+1}{\sqrt{x^2+1}} = a(\sqrt{x^2+1}) + (ax+b)\frac{2x}{2\sqrt{x^2+1}} + \lambda \frac{1}{\sqrt{x^2+1}} \quad \Big| \cdot \sqrt{x^2+1}$$

$$x^2+1 = a(x^2+1) + (ax+b)x + \lambda$$

$$x^2+1 = 2ax^2 + bx + a + \lambda \quad \Rightarrow a = \frac{1}{2} \quad b = 0 \quad \lambda = \frac{1}{2}$$

$$\int \sqrt{x^2+1}\, dx = \frac{1}{2} x \sqrt{x^2+1} + \frac{1}{2} \ln\left|x + \sqrt{x^2+1}\right| + C$$

(3) $$\int \frac{3x+1}{\sqrt{2x^2-x+1}}\, dx = a\sqrt{2x^2-x+1} + \lambda \int \frac{dx}{\sqrt{2x^2-x+1}} \quad \Big| \frac{d}{dx}$$

$$\frac{3x+1}{\sqrt{2x^2-x+1}} = a \frac{4x-1}{2\cdot\sqrt{2x^2-x+1}} + \lambda \frac{1}{\sqrt{2x^2-x+1}} \quad \Big| \cdot 2\sqrt{2x^2-x+1}$$

$2(3x+1) = a(4x-1) + 2\lambda \Rightarrow a = \frac{3}{2} \quad \lambda = \frac{7}{4}$

$$I = \frac{3}{2} \cdot \sqrt{2x^2 - x + 1} + \underbrace{\frac{7}{4} \int \frac{dx}{\sqrt{2x^2 - x + 1}}}_{I_1}$$

$$2x^2 - x + 1 = 2(x^2 - \frac{1}{2}x + \frac{1}{2}) = 2 \cdot (x^2 - 2 \cdot x \cdot \frac{1}{4} + \frac{1}{16} - \frac{1}{16} + \frac{1}{2})$$
$$= 2 \cdot \left[(x - \frac{1}{4})^2 + \frac{7}{16}\right]$$

$$I_1 = \int \frac{dx}{\sqrt{2 \cdot \left[(x - \frac{1}{4})^2 + \frac{7}{16}\right]}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(x - \frac{1}{4})^2 + \frac{7}{16}}} \quad \Big| \; x - \frac{1}{4} = t$$

$$I_1 = \frac{1}{\sqrt{2}} \ln \left| x - \frac{1}{4} + \sqrt{(x - \frac{1}{4})^2 + \frac{7}{16}} \right| + C$$

$$I = \frac{3}{2} \sqrt{2x^2 - x + 1} + \frac{7}{4\sqrt{2}} \ln \left| x - \frac{1}{4} + \sqrt{(x - \frac{1}{4})^2 + \frac{7}{16}} \right| + C$$

za vježbu:

a)* $\int \sqrt{x^2 + 5x + 4} \, dx$

c)* $\int x \sqrt{x^2 + 2x + 2} \, dx$

b)* $\int \frac{2x - 1}{\sqrt{3x - x^2}} \, dx$

d)* $\int \frac{2x + 3}{\sqrt{2x - x^2}} \, dx$

2.

$$\int R(x, \sqrt[n]{ax + b}) \, dx \quad ; \quad \int R\left(x, \sqrt[n]{\frac{ax + b}{cx + d}}\right) dx$$

R - racionalna funkcija

① $I = \int \frac{dx}{\sqrt{2x - 1} - \sqrt[4]{2x - 1}} = \quad \Big| \; \begin{aligned} 2x - 1 &= t^4 \\ 2dx &= 4t^3 dt \; \Big| :2 \\ dx &= 2t^3 dt \end{aligned} \; \Big| = \int \frac{2t^3 dt}{\sqrt{t^4} - \sqrt[4]{t^4}}$

$$= 2\int \frac{t^3}{t^2-t}\,dt = 2\int \frac{\cancel{t^3}^{\,2}}{\cancel{t}(t-1)} = 2\int \frac{t^2-1+1}{t-1}\,dt =$$

$$= 2\left(\int \frac{t^2-1}{t-1}\,dt + \int \frac{1}{t-1}\,dt\right) = 2\left(\int (t+1)\,dt + \ln|t-1|\right)$$

$$= 2\left(\frac{t^2}{2} + t + \ln|t-1|\right) + C = t^2 + 2t + 2\ln|t-1| + C$$

$$= \left(\sqrt[4]{2x-1}\right)^2 + 2\sqrt[4]{2x-1} + 2\ln\left|\sqrt[4]{2x-1} - 1\right| + C$$

$$= \sqrt{2x-1} + 2\sqrt[4]{2x-1} + \ln\left|\left(\sqrt[4]{2x-1} - 1\right)^2\right| + C$$

* (2) $\displaystyle\int \frac{dx}{2\sqrt{x} - \sqrt[3]{x} - \sqrt[4]{x}} = \left| \begin{array}{l} x = t^{12} \\ dx = 12t^{11}\,dt \end{array} \right| = \int \frac{12t^{11}\,dt}{2\sqrt{t^{12}} - \sqrt[3]{t^{12}} - \sqrt[4]{t^{12}}} =$

$$= 12\int \frac{t^{11}\,dt}{2t^6 - t^4 - t^3} = 12\int \frac{t^{11}\,dt}{t^3(2t^3 - t - 1)} = \int \frac{12t^8}{2t^3 - t - 1}\,dt$$

$12t^8 : (2t^3 - t - 1) = 6t^5 \ldots$

$\vdots$

$$2t^3 - t - 1 = t^3 + t^3 - t - 1 = t(t^2-1) + (t^3-1) = t(t-1)(t+1) + (t-1)(t^2+t+1)$$

$$= (t-1)\left[t(t+1) + t^2 + t + 1\right] = (t-1)(2t^2 + 2t + 1) \quad \ldots$$

(3) $\displaystyle I = \int \sqrt{\frac{x+1}{x-1}}\,dx = \left| \begin{array}{ll} \frac{x+1}{x-1} = t^2 \;/\cdot(x-1) & x - t^2x = -t^2 - 1 \;/\cdot(-1) \\ x+1 = t^2(x-1) & t^2x - x = t^2 + 1 \\ x+1 = t^2x - t^2 & x(t^2-1) = t^2+1 \\ & x = \frac{t^2+1}{t^2-1} \end{array} \right.$

$$dx = \frac{2t(t^2-1) - 2t(t^2+1)}{(t^2-1)^2}\,dt$$

$$dx = \frac{-4t\,dt}{(t^2-1)^2}$$

$$J = \int t \cdot \frac{-4t\,dt}{(t^2-1)^2} = \left|\begin{array}{l} u = t \\ du = dt \end{array} \quad \begin{array}{l} dv = \frac{t\,dt}{(t^2-1)^2} \\ v = \int \frac{t\,dt}{(t^2-1)^2} \end{array}\right| \quad \begin{array}{l} t^2-1 = z \\ 2t\,dt = dz \\ t\,dt = \frac{dz}{2} \end{array} \quad \frac{1}{2}\int\frac{dz}{z^2} = \frac{-1}{2z} = \frac{-1}{2(t^2-1)}$$

$$= -4\left[\frac{-t}{2(t^2-1)} + \frac{1}{2}\int\frac{dt}{t^2-1}\right] = \frac{2t}{t^2-1} - 2\cdot\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C$$

$$= \frac{2\sqrt{\frac{x+1}{x-1}}}{\frac{x+1}{x-1}-1} - 2\cdot\ln\left|\frac{\sqrt{\frac{x+1}{x-1}}-1}{\sqrt{\frac{x+1}{x-1}}+1}\right| + C$$

$$= \frac{\cancel{2}\sqrt{\frac{x+1}{x-1}}}{\frac{\cancel{2}}{x-1}} - \ln\left|\frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\right| + C$$

$$= (x-1)\sqrt{\frac{x+1}{x-1}} - \ln\frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}{\left(\sqrt{x+1}\right)^2-\left(\sqrt{x-1}\right)^2}$$

$$= \sqrt{(x+1)(x-1)} - \ln\frac{\left(\sqrt{x+1}-\sqrt{x-1}\right)^2}{2} + C$$

$$= \sqrt{x^2-1} - 2\ln\frac{\left|\sqrt{x+1}-\sqrt{x-1}\right|}{2} + C$$

Za vježbu:

4. $\int \frac{x\,dx}{\sqrt{x+1}+\sqrt[3]{x+1}}$

5. $\int \frac{\sqrt{x+1}+2}{(x+1)^2-\sqrt{x+1}}\,dx$

6. $\int \sqrt{\frac{x}{2-x}}\,dx$

3° Racionalisanje

① $\int \frac{x}{\sqrt{x+2}+\sqrt{x+1}}\,dx = \int \frac{x}{\sqrt{x+2}+\sqrt{x+1}} \cdot \frac{\sqrt{x+2}-\sqrt{x+1}}{\sqrt{x+2}-\sqrt{x+1}}\,dx =$

$= \int \frac{x(\sqrt{x+2}-\sqrt{x+1})}{(\sqrt{x+2})^2-(\sqrt{x+1})^2} = \int \frac{x(\sqrt{x+2}-\sqrt{x+1})}{\cancel{x}+2-\cancel{x}-1} =$

$= \underbrace{\int x\sqrt{x+2}\,dx}_{I_1} - \underbrace{\int x\sqrt{x+1}\,dx}_{I_2}$

$I_1 = \left| \begin{matrix} x+2=t^2 \\ dx=2t\,dt \\ x=t^2-2 \end{matrix} \right| = \int (t^2-2)\cdot t\cdot 2t\,dt = 2\int (t^4-2t^2)\,dt =$

$= 2\left(\frac{t^5}{5}-\frac{2t^3}{3}\right)+C = 2\cdot t^3\left(\frac{t^2}{5}-\frac{2}{3}\right)+C =$

$= 2(\sqrt{x+2})^3\left(\frac{x+2}{5}-\frac{2}{3}\right)+C = 2\cdot(\sqrt{x+2})^2\sqrt{x+2}\ \frac{3x+6-10}{15}+C$

$= 2(x+2)\sqrt{x+2}\cdot\frac{3x-4}{15}+C$

$I_2 = \left| \begin{matrix} x+1=t^2 \\ dx=2t\,dt \\ x=t^2-1 \end{matrix} \right| = \int (t^2-1)\cdot t\cdot 2t\,dt = 2\int (t^4-t^2)\,dt =$

$= 2\left(\frac{t^5}{5}-\frac{t^3}{3}\right)+C = 2t^3\left(\frac{t^2}{5}-\frac{1}{3}\right)+C = 2(\sqrt{x+1})^3\left(\frac{x+1}{5}-\frac{1}{3}\right)+C$

$= 2(x+1)\sqrt{x+1}\ \frac{3x-2}{15}+C$

$I = \frac{2}{15}\left[(x+2)(3x-4)\sqrt{x+2}-(x+1)(3x-2)\sqrt{x+1}\right]+C$

za vježbu:

(2.) $\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,dx$

(3.) $\int \frac{dx}{x-\sqrt{x^2-1}}$.

4° $\int \frac{Mx+N}{(x-\alpha)^n\sqrt{ax^2+bx+c}}\,dx \quad (n\in\mathbb{N},\ M,N,a,b,c,\alpha\in\mathbb{R},\ a\neq 0)$

smjena: $x-\alpha=\frac{1}{t}$

(1) $I=\int\frac{dx}{(x+1)\sqrt{x^2+x+1}} = \left| \begin{matrix} x+1=\frac{1}{t} \Rightarrow x=\frac{1}{t}-1 \\ dx=-\frac{1}{t^2}\,dt \end{matrix} \right| =$

$= \int \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{(\frac{1}{t}-1)^2+\frac{1}{t}-\cancel{1}+1}} = -\int\frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{1}{t^2}-\frac{2}{t}+1+\frac{1}{t}}}$

$= -\int \frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\frac{t^2-t+1}{t^2}}} = -\int\frac{\frac{1}{t^2}\,dt}{\frac{1}{t}\frac{\sqrt{t^2-t+1}}{t}} = -\int\frac{dt}{\sqrt{t^2-t+1}}$

$= -\int\frac{dt}{\sqrt{(t-\frac{1}{2})^2+\frac{3}{4}}} = \left| \begin{matrix} t-\frac{1}{2}=z \\ \dots \end{matrix} \right| = -\ln\left|t-\frac{1}{2}+\sqrt{t^2-t+1}\right|+C, \quad t=\frac{1}{x+1}$

(2) $I=\int\frac{dx}{x^3\sqrt{x^2+1}} = \left| \begin{matrix} x=\frac{1}{t} \\ dx=-\frac{1}{t^2}\,dt \end{matrix} \right| =$

$= \int\frac{-\frac{1}{t^2}\,dt}{\frac{1}{t^3}\sqrt{\frac{1}{t^2}+1}} = -\int\frac{\frac{1}{t^2}\,dt}{\frac{1}{t^3}\sqrt{\frac{1+t^2}{t^2}}} = -\int\frac{\frac{1}{t^2}\,dt}{\frac{1}{t^3}\frac{\sqrt{1+t^2}}{t}} = -\int\frac{t^2}{t^3\sqrt{1+t^2}}\,dt$

$= -\int\frac{t^2}{\sqrt{t^2+1}}\,dt = \int\frac{-t^2}{\sqrt{t^2+1}} = (at+b)\sqrt{t^2+1}+\lambda\int\frac{dt}{\sqrt{t^2+1}} \quad \Big| \frac{d}{dt}$

$$= \frac{-t^2}{\sqrt{t^2-1}} = a\sqrt{t^2+1} + (at+b)\cdot\frac{2t}{2\sqrt{t^2+1}} + \lambda\frac{1}{\sqrt{t^2+1}} \quad /\cdot\sqrt{t^2+1}$$

$$-t^2 = a(t^2+1) + (at+b)\cdot t + \lambda$$

$$-t^2 = 2at^2 + bt + a + \lambda$$

$a = \frac{1}{2}$

$b = 0$

$\lambda = \frac{1}{2}$

$$y = -\frac{1}{2} t\sqrt{t^2+1} + \frac{1}{2}\ln\left|t+\sqrt{t^2+1}\right| + C$$

$$y = -\frac{1}{2}\,\frac{1}{x}\sqrt{\frac{1}{x^2}+1} + \frac{1}{2}\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}+1}\right| + C$$

$$y = -\frac{\sqrt{1+x^2}}{2x^2} + \frac{1}{2}\ln\left|\frac{1+\sqrt{1+x^2}}{x}\right| + C$$

II način

$$\int\frac{dx}{x^3\sqrt{x^2+1}} \quad \frac{/\cdot x}{/\cdot x} = \int\frac{x\,dx}{x^4\sqrt{x^2+1}} = \left| \begin{array}{l} x^2+1 = t^2 \Rightarrow x^2 = t^2-1 \\ 2x\,dx = 2t\,dt \\ x\,dx = t\,dt \end{array} \right.$$

$$= \int\frac{\cancel{t}\,dt}{(t^2-1)^2\cdot\cancel{t}} = \int\frac{dt}{(t^2-1)^2} = \int\frac{dt}{(t-1)^2(t+1)^2}$$

$$\frac{1}{(t-1)^2(t+1)^2} = \frac{a}{t-1} + \frac{b}{(t-1)^2} + \frac{c}{t+1} + \frac{d}{(t+1)^2}$$

Za vježbu:

3. $$\int \frac{dx}{(x-1)^3\sqrt{x^2+3x+1}}$$ $(x-1)=\frac{1}{t}$

4. $$\int \frac{dx}{x^2\sqrt{x^2+x+1}}$$ $x=\frac{1}{t}$

5. $$\int \frac{(3x+2)\,dx}{(x+1)\cdot\sqrt{x^2+3x+3}}$$ $x+1=\frac{1}{t}$

5° Integracija binomnog diferencijala

$$\int x^m (a+bx^n)^p\,dx \qquad m,n,p \in \mathbb{Q}$$

1° $p \in \mathbb{Z} \Rightarrow$ smjena: $x=t^s$

s - NZS za nazivnike m i n

2° $\frac{m+1}{n} \in \mathbb{Z} \Rightarrow$ smjena: $a+bx^n=t^s$

s - nazivnik broja p

3° $\frac{m+1}{n}+p \in \mathbb{Z} \Rightarrow$ smjena: $ax^{-n}+b=t^s$

s - nazivnik broja p

Napomena: $(a+bx^n)\cdot x^{-n} = ax^{-n}+b$

ili: $ax^{-n}+b=t^s \quad | \cdot x^n$

$a+bx^n=t^s\cdot x^n$

$m = -\frac{3}{4}$ NZS$(4,6) = 12$ 1 slučaj

$n = \frac{1}{6}$

(1) $\int x^{-\frac{3}{4}}(1+x^{\frac{1}{6}})^{-1}dx = \Big| p=-1 \quad x = t^{12} \Rightarrow dx = 12t^{11}dt \Big|$

$= \int (t^{12})^{-\frac{3}{4}} \cdot (1+(t^{12})^{\frac{1}{6}})^{-1} \cdot 12t^{11}dt = 12\int t^{-9}(1+t^2)^{-1} \cdot t^{11}dt$

$= 12 \int \frac{t^2+1-1}{1+t^2}dt = 12\left(\int \frac{t^2+1}{1+t^2}dt - \int \frac{1}{1+t^2}dt\right) =$

$= 12(t - \operatorname{arctg} t) + C = 12(\sqrt[12]{x} - \operatorname{arctg}\sqrt[12]{x}) + C$

(2) $\int \frac{\sqrt{1+\sqrt[3]{x}}}{\sqrt[3]{x^2}}dx = \int \frac{\sqrt{(1+x^{\frac{1}{3}})}}{x^{\frac{2}{3}}}dx = \int x^{-\frac{2}{3}}(1+x^{\frac{1}{3}})^{\frac{1}{2}}dx = \Big| m=-\frac{2}{3},\ n=\frac{1}{3},\ p=\frac{1}{2} \Big|$

$\frac{m+1}{n} = \frac{\frac{1}{3}}{\frac{1}{3}} = 1 \in \mathbb{Z}$ 2 slučaj: $1+x^{\frac{1}{3}} = t^2$, $\frac{1}{3}x^{-\frac{2}{3}}dx = 2tdt \ | \cdot 3$, $x^{-\frac{2}{3}}dx = 6tdt$ $\Big| = \int (t^2)^{\frac{1}{2}} \cdot 6tdt = 6\int t^2 dt$

$= 6 \cdot \frac{t^3}{3} + C = 2t^3 + C$

$= 6 \cdot (\sqrt{1+\sqrt[3]{x}})^3 + C$

(3) $\int \frac{dx}{x^2\sqrt{(1+x^2)^3}} = \int x^{-2} \cdot (1+x^2)^{-\frac{3}{2}}dx = \Big| m=-2,\ n=2,\ p=-\frac{3}{2} \quad \frac{m+1}{n} = -\frac{1}{2} \notin \mathbb{Z}$

$\frac{m+1}{n} + p = -\frac{1}{2} - \frac{3}{2} = -2 \Rightarrow$ $x^{-2}+1 = t^2$, $x^{-2} = t^2 - 1 \ )^{-\frac{1}{2}}$, $x = (t^2-1)^{-\frac{1}{2}}$ $dx = -\frac{1}{2}(t^2-1)^{-\frac{3}{2}} \cdot 2tdt$, $dx = -t(t^2-1)^{-\frac{3}{2}}dt \Big|$

3. slučaj

$= \int (t^2-1)\left(1+\frac{1}{t^2-1}\right)^{-\frac{3}{2}} \cdot (-t)(t^2-1)^{-\frac{3}{2}} \cdot dt =$

$= \int (t^2-1)\left(\frac{t^2-1+1}{t^2-1}\right)^{-\frac{3}{2}} \cdot (-t)(t^2-1)^{-\frac{3}{2}}dt = -\int (t^2-1) \cdot \frac{(t^2)^{-\frac{3}{2}}}{(t^2-1)^{-\frac{3}{2}}} \cdot (t^2-1)^{-\frac{3}{2}} \cdot t\,dt$

$= -\int t^{-2}(t^2-1)\,dt = -\int (1-t^{-2})\,dt = -\int dt + \int t^{-2}\,dt =$

$= -t + \frac{t^{-1}}{-1} + C = -t - \frac{1}{t} + C = -\sqrt{x^2+1} - \frac{1}{\sqrt{x^2+1}} + C$

*④ $\int \frac{dx}{\sqrt[6]{(1+x^6)^7}} = \int \sqrt[6]{(1+x^6)^{-7}}\,dx = \int (1+x^6)^{-\frac{7}{6}} = \left| \begin{array}{l} m=0 \\ n=6 \\ p=-\frac{7}{6} \end{array} \quad \frac{1}{6} + \frac{-7}{6} = -1 \quad 3.\ \text{slučaj} \right.$

$(x^{-6}+1) = t^6 \qquad dx = -\frac{1}{6}\cdot(t^6-1)^{-\frac{7}{6}}\,6\cdot t^5\,dt$

$x^{-6} = t^6 - 1 \quad / ^{-\frac{1}{6}}$

$x = (t^6-1)^{-\frac{1}{6}} \qquad dx = -t^5(t^6-1)^{-\frac{7}{6}}\,dt$

$= \int \left(1 + \frac{1}{t^6-1}\right)^{-\frac{7}{6}} \cdot (-t^5)\cdot(t^6-1)^{-\frac{7}{6}}\,dt =$

$-\int t^5 \frac{(t^6)^{-\frac{7}{6}}}{(t^6-1)^{-\frac{7}{6}}}\cdot(t^6-1)^{-\frac{7}{6}}\,dt = -\int t^5\cdot t^{-7}\,dt = -\int t^{-2}\,dt$

$= \frac{1}{t} + C = \frac{1}{\sqrt[6]{x^{-6}+1}} + C$

za vježbu:

*⑤ $\int \frac{dx}{\sqrt{x^3}\cdot\sqrt[3]{1+\sqrt[4]{x^3}}}$

⑦ $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\,dx$

*⑥ $\int \frac{\sqrt{1+x^4}}{x^5}\,dx$

⑧ $\int \frac{dx}{x^3\sqrt[3]{2-x^3}}$

6. Eulerove smjene

$\int R\left(x, \sqrt{ax^2+bx+c}\right) dx$, R - racionalna funkcija

1° $\sqrt{ax^2+bx+c} = \pm\sqrt{a}\, x + t$ ako je $a>0$

2° $\sqrt{ax^2+bx+c} = xt \pm \sqrt{c}$, ako je $c>0$

3° $\sqrt{ax^2+bx+c} = \sqrt{a(x-x_1)(x-x_2)} = t(x-x_1)$ (ili $t(x-x_2)$)

ako su $x_1, x_2 \in \mathbb{R}$

① $I = \int \frac{dx}{x+\sqrt{x^2+x+1}}$

smjena: $\sqrt{x^2+x+1} = -x+t \quad (\Rightarrow x+\sqrt{x^2+x+1} = t) \quad |^2$

$x^2+x+1 = (t-x)^2$

$\cancel{x^2}+x+1 = t^2-2tx+\cancel{x^2}$ $\qquad x+2tx = t^2-1 \qquad x = \frac{t^2-1}{1+2t}$

$x(1+2t) = t^2-1$

$dx = \frac{2t(1+2t)-(t^2-1)\cdot 2}{(1+2t)^2} dt$

$dx = \frac{2t^2+2t+2}{(1+2t)^2} dt$

$dx = \frac{2t+4t^2-2t^2+2}{(1+2t)^2} dt$

$I = \int \frac{\frac{2t^2+2t+2}{(1+2t)^2}}{t} = \int \frac{2t^2+2t+2}{t(1+2t)^2}$

$\frac{2t^2+2t+2}{t(1+2t)^2} = \frac{a}{t} + \frac{b}{1+2t} + \frac{c}{(1+2t)^2}$

$2t^2+2t+2 = a(1+2t)^2 + bt(1+2t) + ct$

za $t=0 \Rightarrow a=2$

za $t=-\frac{1}{2} \Rightarrow c=-3$

za $t=1 \Rightarrow b=-3$

$$I=\int\left(\frac{2}{t}-\frac{3}{1+2t}-\frac{3}{(1+2t)^2}\right)dt=$$

$$=2\int\frac{dt}{t}-3\int\frac{dt}{1+2t}-3\int\frac{dt}{(1+2t)^2}=$$

$$=2\ln|t|-\frac{3}{2}\ln|1+2t|+\frac{3}{2(1+2t)}+C$$

$$=2\ln|x+\sqrt{x^2+x+1}|-\frac{3}{2}\ln|1+2x+2\sqrt{x^2+x+1}|+\frac{3}{2(1+2x+2\sqrt{x^2+x+1})}+C$$

(2) $$I=\int\frac{dx}{1+\sqrt{1-2x-x^2}}$$

$$\sqrt{1-2x-x^2}=xt-1 \quad |^2$$

$$1-2x-x^2=(xt-1)^2$$

$$\cancel{1}-2x-x^2=x^2t^2-2xt+\cancel{1}$$

$$-2x-x^2=x^2t^2-2xt \quad |:x$$

$$-2-x=xt^2-2t$$

$$-2+2t=xt^2+x$$

$$-2+2t=x(t^2+1)$$

$$x=\frac{2t-2}{t^2+1}$$

$$dx=\frac{2(t^2+1)-(2t-2)2t}{(t^2+1)^2}dt$$

$$dx=\frac{2t^2+2-4t^2+4t}{(t^2+1)^2}dt$$

$$dx=\frac{-2t^2+4t+2}{(t^2+1)^2}dt$$

$$I=\int\frac{\frac{-2t^2+4t+2}{(t^2+1)^2}}{\frac{2t-2}{t^2+1}\cdot t}dt=\int\frac{-2t^2+4t+2}{t(2t-2)(t^2+1)}dt$$

$$I=\int\frac{-t^2+2t+1}{t(t-1)(t^2+1)}dt$$

$$\frac{-t^2+2t+1}{t(t-1)(t^2+1)} = \frac{a}{t} + \frac{b}{t-1} + \frac{ct+d}{t^2-1} \ldots$$

$a=-1, \quad b=1, \quad c=0, \quad d=2$

$$\boxed{I} = \int\left(-\frac{1}{t} + \frac{1}{t-1} + \frac{2}{t^2-1}\right)dt =$$

$$I = -\int\frac{dt}{t} + \int\frac{1}{t-1}\,dt + \int\frac{2}{t^2+1}\,dt = -\ln|t| + \ln|t-1| + 2\operatorname{arctg} t + C$$

$$\sqrt{1-2x-x^2} = xt-1 \quad \ldots \quad t = \frac{\sqrt{1-2x-x^2}+1}{x}$$

uvrstiti umjesto t!

$$③ \int \frac{x-\sqrt{x^2+3x+2}}{x+\sqrt{x^2+3x+2}}\,dx =$$

$x^2+3x+2 = (x+1)(x+2)$

$\sqrt{(x+1)(x+2)} = t(x+1) \quad /^2$

$(x+1)(x+2) = t^2(x+1)^2 \quad /:(x+1)$

$x+2 = t^2(x+1) \quad \Rightarrow \quad x+2 = t^2x+t^2$

$x - t^2x = t^2-2$

$x(1-t^2) = t^2-2 \quad \Rightarrow \quad x = \frac{t^2-2}{1-t^2}$

$$dx = \frac{2t(1-t^2)-(t^2-2)\cdot(-2t)}{(1-t^2)^2}\,dt$$

$$dx = \frac{2t-2t^3+2t^3-4t}{(1-t^2)^2}\,dt = \frac{-2t}{(1-t^2)^2}\,dt$$

$$\sqrt{x^2+3x+2} = t(x+1) = t\cdot\left(\frac{t^2-2}{1-t^2}+1\right) = t\cdot\frac{\cancel{t^2}-2+1-\cancel{t^2}}{1-t^2} = \frac{-t}{1-t^2}$$

$$\int \frac{\frac{t^2-2}{\cancel{1-t^2}} + \frac{t}{\cancel{1-t^2}}}{\frac{t^2-2}{\cancel{1-t^2}} - \frac{t}{\cancel{1-t^2}}} \cdot \frac{-2t\,dt}{(1-t^2)^2} = -2\int \frac{t\cdot(t^2+t-2)}{(t^2-t-2)(1-t^2)^2}\,dt$$

$t^2+t-2 = t^2+2t-t-2 = t(t+2)-(t+2) = (t+2)(t-1)$

$t^2-t-2 = t^2+t-2t-2 = t(t+1)-2(t+1) = (t-2)(t+1)$

$$2\int \frac{t(t+2)(t-1)\,dt}{(t-2)(t+1)(t-1)^2(t+1)^2} = 2\int \frac{t^2+2t}{(t-2)(t-1)(t+1)^3}\,dt$$

Za vježbu:

(4) $\int \frac{1-\sqrt{x^2+x+1}}{x\sqrt{x^2+x+1}}\,dx$

(5) $\int x\sqrt{x^2-2x+2}\,dx$

(6) $\int \frac{dx}{(1+\sqrt{x+x^2})^2}$

Integracija trigonometrijskih funkcija

I tip: $\int \sin^m x \cdot \cos^n x\,dx \quad (m,n \in \mathbb{N} \cup \{0\})$

a) m ili n je neparan

$\Rightarrow \sin x = t$ ili $\cos x = t$

(1) $\int \sin^3 x \cdot \cos^4 x\,dx = \int \sin x \cdot \sin^2 x \cdot \cos^4 x\,dx =$

$= \int \sin x\,(1-\cos^2 x)\cdot \cos^4 x\,dx = \left| \begin{array}{l} \cos x = t \\ -\sin x\,dx = dt \\ \sin x\,dx = -dt \end{array} \right| = \int (1-t^2)\cdot t^4\,(-dt) =$

$= \int (-t^4+t^6)\,dt = -\frac{t^5}{5} + \frac{t^7}{7} + C = -\frac{\cos^5 x}{5} + \frac{\cos^7 x}{7} + C$

(2) $\int \sin^2 x \cos^5 x \, dx = \int \cos x \cdot \cos^4 x \, \sin^2 x \, dx = \int \cos x \, (1+\cos^2 x)^2 \cdot \sin^2 x \, dx$

$\left[\begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right] = \int (1-t^2)^2 \, t^2 \, dt = \int (1-2t^2+t^4) \, t^2 \, dt =$

$= \int t^2 \, dt - 2\int t^4 \, dt + \int t^6 \, dt = \frac{t^3}{3} - 2 \cdot \frac{t^5}{5} + \frac{t^7}{7} + C =$

$= \frac{\sin^3 x}{3} - 2\frac{\sin^5 x}{5} + \frac{\sin^7 x}{7} + C$

(3) $\int \frac{\cos^5 x}{\sin^2 x} \, dx = \int \frac{\cos x \, (1-\sin^2 x)^2}{\sin^2 x} \, dx \quad \left[\begin{array}{l} \sin x = t \\ \cos x \, dx = dt \end{array} \right] =$

$= \int \frac{(1-t^2)^2}{t^2} \, dt = \int \frac{1-2t^2+t^4}{t^2} \, dt = \int \frac{1}{t^2} \, dt - 2\int \frac{t^2}{t^2} \, dt + \int \frac{t^4}{t^2} \, dt$

$= -\frac{1}{t} - 2t + \frac{t^3}{3} + C = -\frac{1}{\sin x} - 2\sin x + \frac{\sin^3 x}{3} + C$

Za vježbu:

(4) $\int \sin^5 x \cdot \cos^6 x \, dx$

(5) $\int \cos^7 x \, dx$ s Amelkom

(6) $\int \frac{\sin^3 x}{\cos^4 x} \, dx$

b) m i n su parni

načini: 1° parcijalna integracija

2° $\sin^2 x = \frac{1-\cos 2x}{2}$, $\cos^2 x = \frac{1+\cos 2x}{2}$

3° $\sin x = \frac{e^{ix}-e^{-ix}}{2i}$, $\cos x = e^{ix}+e^{-ix}$

$(e^{ix} = \cos x + i\sin x \; ; \; e^{-ix} = \cos x - i\sin x)$.

① $\int \cos^4 x\,dx \overset{2^\circ}{=} \int \left(\frac{1+\cos 2x}{2}\right)^2 dx = \int \frac{1+2\cos 2x+\cos^2 2x}{4}\,dx$

$= \frac{1}{4}\left(\int dx + 2\int \cos 2x\,dx + \int \cos^2 2x\,dx\right) =$ $\left\| \begin{array}{l} \int \cos \alpha x\,dx = \frac{1}{\alpha}\sin \alpha x + C \\ \int \sin \alpha x\,dx = -\frac{1}{\alpha}\cos \alpha x + C \end{array} \right\|$

$= \frac{1}{4}\left(x + 2\cdot\frac{1}{2}\sin 2x + \int \frac{1+\cos 4x}{2}\,dx\right) =$

$= \frac{1}{4}\left[x + \sin 2x + \frac{1}{2}\left(\int dx + \int \cos 4x\,dx\right)\right] =$

$= \frac{1}{4}\left[x + \sin 2x + \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right)\right] + C = \frac{1}{4}\left(\frac{3x}{2} + \sin 2x + \frac{1}{8}\sin 4x\right) + C$

$= \frac{3x}{8} + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$

ili:

$\cos^4 x = \left(\frac{e^{ix}+e^{-ix}}{2}\right)^4 = \frac{e^{4ix} + 4\cdot e^{3ix}\cdot e^{-ix} + 6\cdot e^{2ix}\cdot e^{-2ix} + 4\cdot e^{ix}\cdot e^{-3ix} + e^{-4ix}}{16}$

$(a+b)^4 = a^4 + 4a^3 b + 6a^2 b^2 + 4ab^3 + b^4$

$= \frac{(e^{4ix}+e^{-4ix}) + 4\cdot(e^{2ix}+e^{-2ix}) + 6}{16} =$ $\left\| e^{i\alpha} + e^{-i\alpha} = 2\cos\alpha \right\|$

$= \frac{2\cos 4x + 4\cdot 2\cos 2x + 6}{16} \begin{array}{l}:2\\:2\end{array} = \frac{\cos 4x + 4\cos 2x + 3}{8}$

$I = \int \cos^4 x\,dx = \int \frac{\cos 4x + 4\cos 2x + 3}{8}\,dx =$

$= \frac{1}{8}\left(\int \cos 4x\,dx + 4\int \cos 2x\,dx + 3\int dx\right) =$

$= \frac{1}{8}\left(\frac{1}{4}\sin 4x + 4\cdot\frac{1}{2}\sin 2x + 3\cdot x\right) + C = \frac{1}{32}\sin 4x + \frac{1}{4}\sin 2x + \frac{3x}{8} + C$

(2) $I=\int \sin^2 x \cdot \cos^2 x \, dx = \int \frac{1-\cos 2x}{2} \cdot \frac{1+\cos 2x}{2} \, dx =$

$= \frac{1}{4} \int (1-\cos 2x)(1+\cos 2x) \, dx = \frac{1}{4} \int 1 - \cos^2 2x =$

$= \frac{1}{4} \int dx - \frac{1}{4} \int \cos^2 2x \, dx = \frac{1}{4} \left(x - \int \frac{1+\cos 4x}{2} \, dx \right) =$

$= \frac{1}{4} \left[x - \frac{1}{2} \left(\int dx + \int \cos 4x \, dx \right) \right] = \frac{1}{4} x - \frac{1}{8} x - \frac{1}{32} \sin 4x + C =$

$= \frac{1}{8} x - \frac{1}{32} \sin 4x + C$

II način:

$\sin x \cdot \cos x = \frac{1}{2} \sin 2x$

$I = \int \frac{1}{4} \sin^2 2x \, dx = \frac{1}{4} \int \frac{1-\cos 4x}{2} \, dx = \ldots$

za vježbu:

(3) $\int \sin^4 x \cos^2 x \, dx$

(4) $\int \sin^2 x \cdot \cos^4 x \, dx$

(5) $\int \sin^6 x \, dx$

29.03. 2010

II tip: $\int \sin \alpha x \cdot \cos \beta x \, dx$, $\int \sin \alpha x \sin \beta x \, dx$, $\int \cos \alpha x \cos \beta x \, dx$

$$\sin \alpha \cdot \cos \beta = \frac{1}{2} [\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$
$$\sin \alpha \cdot \sin \beta = \frac{1}{2} [\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$
$$\cos \alpha \cdot \cos \beta = \frac{1}{2} [\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\int \sin \alpha x \, dx = -\frac{1}{\alpha} \cos \alpha x + C$$

$$\int \cos \alpha x \, dx = \frac{1}{\alpha} \sin \alpha x + C$$

① $I = \int \sin 2x \cos 3x \, dx = \frac{1}{2} \int [\sin 5x + \sin(-x)] \, dx$
$= \frac{1}{2}\left(\int \sin 5x \, dx - \int \sin x \, dx\right) = \frac{1}{2}\left(-\frac{1}{5}\cos 5x + \cos x\right) + C$
$= -\frac{1}{10} \cos 5x + \frac{1}{2} \cos x + C$

② $\int \sin x \sin 2x \sin 3x \, dx = \frac{1}{2} \int [\cos(-x) - \cos 3x] \cdot \sin 3x \, dx$

$= \frac{1}{2}\left[\int \cos x \cdot \sin 3x \, dx - \int \cos 3x \sin 3x \, dx\right]$

$= \frac{1}{2}\left[\frac{1}{2}\int (\sin 4x + \sin 2x) \, dx - \frac{1}{2}\int (\sin 0 + \sin 6x) \, dx\right] dx =$

$= \frac{1}{4}\left(\int \sin 4x \, dx + \int \sin 2x \, dx - \int \sin 6x \, dx\right) =$

$= \frac{1}{4}\left(-\frac{1}{4}\cos 4x - \frac{1}{2}\cos 2x + \frac{1}{6}\cos 6x\right) + C = -\frac{1}{16}\cos 4x - \frac{1}{8}\cos 2x + \frac{1}{24}\cos 6x + C$

③ $\int \sin^2 3x \cdot \cos 4x \, dx = \int \frac{1 - \cos 6x}{2} \cos 4x \, dx =$

$= \frac{1}{2} \int (\cos 4x - \cos 6x \cdot \cos 4x) \, dx =$

$= \frac{1}{2}\int\left(\cos 4x\,dx - \int \cos 6x \cdot \cos 4x\,dx\right) =$

$= \frac{1}{2}\left[\frac{1}{4}\sin 4x - \frac{1}{2}\int(\cos 10x + \cos 2x)\,dx\right] =$

$= \frac{1}{8}\sin 4x - \frac{1}{4}\left(\int \cos 10x\,dx + \int \cos 2x\,dx\right) =$

$= \frac{1}{8}\sin 4x - \frac{1}{4}\left(\frac{1}{10}\sin 10x + \frac{1}{2}\sin 2x\right) + C =$

$= \frac{1}{8}\sin 4x - \frac{1}{40}\sin 10x - \frac{1}{8}\sin 2x + C$

Za vježbu:

3. $\int \cos 2x \cos 3x \cos 4x\,dx$
4. $\int \sin^2 2x \cdot \cos^3 x\,dx$
5. $\int \sin^4 3x \cdot \cos^2 2x\,dx$

III tip: $\int R(\sin x, \cos x)\,dx$

R- racionalna funkcija

* univerzalna smjena: $tg\frac{x}{2} = t \Rightarrow \begin{cases} dx = \frac{2dt}{1+t^2} \\ \sin x = \frac{2t}{1+t^2} \\ \cos x = \frac{1-t^2}{1+t^2} \end{cases}$

① $\int \frac{dx}{\sin x} = \left| tg\frac{x}{2} = t \right| = \int \frac{\frac{2dt}{\cancel{1+t^2}}}{\frac{\cancel{2}t}{\cancel{1+t^2}}} = \int \frac{dt}{t} = \ln|t| + C$

$= \ln\left|tg\frac{x}{2}\right| + C$

② $\int \frac{dx}{5 - 4\sin x + 3\cos x} = \left| tg\frac{x}{2} = t \right| = \int \frac{\frac{2dt}{1+t^2}}{5 - 4\cdot\frac{2t}{1+t^2} + 3\cdot\frac{1-t^2}{1+t^2}} =$

$= 2\int \frac{\frac{dt}{\cancel{1+t^2}}}{\frac{5(1+t^2) - 4t + 3(1-t^2)}{\cancel{1+t^2}}} = 2\int \frac{dt}{5 + 5t^2 - 8t + 3 - 3t^2} = \cancel{2}\cdot\int \frac{dt}{\cancel{2}(t^2 - 4t + 4)}$

$$= \int \frac{dt}{(t-2)^2} = -\frac{1}{t-2} + C = -\frac{1}{tg\frac{x}{2}-2} + C$$

③* $$\int \frac{2-\sin x}{2+\cos x}\,dx = \left| tg\frac{x}{2} = t \right| = \int \frac{2-\frac{2t}{1+t^2}}{2+\frac{1-t^2}{1+t^2}} \cdot \frac{2dt}{1+t^2} =$$

$$= \int \frac{\frac{2(1+t^2)-2t}{\cancel{1+t^2}}}{\frac{2(1+t^2)+1-t^2}{\cancel{1+t^2}}} \cdot \frac{2dt}{1+t^2} = \int \frac{2+2t^2-2t}{2+2t^2+1-t^2} \cdot \frac{2dt}{1+t^2} =$$

$$= \int \frac{2(t^2-t+1)}{t^2+3} \cdot \frac{2dt}{1+t^2} = 4\int \frac{t^2-t+1}{(t^2+1)(t^2+3)}\,dt$$

$$\frac{t^2-t+1}{(1+t^2)(t^2+3)} = \frac{at+b}{1+t^2} + \frac{ct+d}{t^2+3} \quad \Big| \, (1+t^2)(t^2+3)$$

$$t^2-t+1 = (at+b)(t^2+3) + (ct+d)(1+t^2)$$
$$t^2-t+1 = at^3 + 3at + bt^2 + 3b + ct + ct^3 + d + dt^2$$
$$t^2-t+1 = t^3(a+c) + t^2(b+d) + t(3a+c) + (3b+d)$$

$a+c=0 \Rightarrow a = -\frac{1}{2}$

$b+d=1 \Rightarrow b = 0$

$3a+c=1 \Rightarrow c = \frac{1}{2}$

$3b+d=1 \Rightarrow d = 1$

$$J = 4\int \left(\frac{-\frac{1}{2}t}{1+t^2} + \frac{\frac{1}{2}t+1}{t^2+3} \right) dt = -2\int \frac{t\,dt}{1+t^2} + \int \frac{2t+4}{t^2+3}\,dt =$$

$$= -\ln(1+t^2) + \int \frac{2t}{t^2+3}\,dt + 4\int \frac{dt}{t^2+3} =$$

$$= -\ln(1+t^2) + \ln(t^2+3) + 4\frac{1}{\sqrt{3}}\,arctg\frac{t}{\sqrt{3}} + C \quad - \quad t = tg\frac{x}{2}$$

$$I = \ln \frac{t^2+3}{t^2+1} + \frac{4}{\sqrt{3}} \text{arctg} \frac{t}{\sqrt{3}} + C$$

Za vježbu:

* (4.) $\displaystyle\int \frac{dx}{8-4\sin x+7\cos x}$

* (5.) $\displaystyle\int \frac{1-\sin x+\cos x}{1+\sin x-\cos x}\,dx$

(6) $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x+\cos^3 x}$

IV tip $\int R(\sin^2 x, \sin x\cos x, \cos^2 x)\,dx$

R- racionalna funkcija

smjena: $\quad tg x = t$

$$\left\{ \begin{aligned} dx &= \frac{dt}{1+t^2} \\ \sin^2 x &= \frac{t^2}{1+t^2} \\ \cos^2 x &= \frac{1}{1+t^2} \\ \sin x\cos x &= \frac{t}{1+t^2} \end{aligned} \right.$$

$$\int R(tg x)\,dx$$

* (1) $\displaystyle\int \frac{dx}{\sin^2 x-4\sin x\cos x+5\cos^2 x} = \left| tg x = t \right| =$

$$= \int \frac{\frac{dt}{1+t^2}}{\frac{t^2}{1+t^2}-\frac{4t}{1+t^2}+\frac{5}{1+t^2}} = \int \frac{dt}{t^2-4t+5}\,dt =$$

$$= \int \frac{dt}{(t-2)^2+1} = \text{arctg}(t-2)+C = \text{arctg}(tg x-2)+C$$

$$\boxed{tg\,x = t}$$

② $$\int \frac{dx}{\cos^4 x} = \int \frac{\frac{dt}{1+t^2}}{\left(\frac{1}{1+t^2}\right)^2} = \int (t^2+1)\,dt = \frac{t^3}{3} + t + C =$$

$$= \frac{tg^3 x}{3} + tg\,x + C$$

③ $$I = \int \frac{\cos x + 2\sin x}{4\cos x + 3\sin x}\,dx = \frac{/\cos x}{/\cos x}\int \frac{\frac{\cos x}{\cos x} + \frac{2\sin x}{\cos x}}{\frac{4\cos x}{\cos x} + \frac{3\sin x}{\cos x}} =$$

$$= \int \frac{1 + 2tg\,x}{4 + 3tg\,x}\,dx = \int \frac{1+2t}{4+3t}\,\frac{dt}{1+t^2}$$

$$\frac{1+2t}{(4+3t)(1+t^2)} = \frac{a}{4+3t} + \frac{bt+c}{1+t^2} \;\ldots\; \text{itd}$$

II način:

$$(4\cos x + 3\sin x)' = -4\sin x + 3\cos x$$

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + C$$

$$\cos x + 2\sin x = \alpha(4\cos x + 3\sin x) + \beta(-4\sin x + 3\cos x)$$
$$= (4\alpha + 3\beta)\cos x + (3\alpha - 4\beta)\sin x \Rightarrow$$

$$4\alpha + 3\beta = 1 \quad\Rightarrow\quad \alpha = \frac{2}{5}$$
$$3\alpha - 4\beta = 2 \quad\Rightarrow\quad \beta = -\frac{1}{2}$$

$$I = \int \frac{\frac{2}{5}(4\cos x + 3\sin x) - \frac{1}{5}(-4\sin x + 3\cos x)}{4\cos x + 3\sin x}\,dx$$

$$= \frac{2}{5}\int dx - \frac{1}{5}\int \frac{-4\sin x + 3\cos x}{4\cos x + 3\sin x}\,dx =$$

$= \frac{2}{5}x - \frac{1}{5}\ln|4\cos x + 3\sin x| + C$

Za vježbu:

(4.) $\int tg^3 x\, dx$

(5.) $\int \frac{dx}{\sin^4 x}$

(6) $\int \frac{dx}{3\cos^2 x + 4\sin^2 x}$

(7) $\int \frac{tg\, x}{tg^2 x - 2tg\, x - 3}\, dx$

(8) $\int \frac{\sin x}{\sin x + \cos x}\, dx$

$\overline{\text{V}}$ tip: rješavanje integrala iracionalnih funkcija pomoću trigonometrijskih smjena

nije + nego −

$\int f(\sqrt{a^2+x^2})\, dx \longrightarrow$ smjena $x = a\sin t$

$a^2 - x^2 = a^2 - a^2\sin^2 t = a^2(1-\sin^2 t) = a^2\cos^2 t$

$\int f(\sqrt{a^2+x^2})\, dx \longrightarrow$ smjena: $x = a\, tg\, t$

$a^2 + x^2 = a^2 + a^2 tg^2 t = a^2\left(1 + \frac{\sin^2 t}{\cos^2 t}\right) = a^2 \frac{\cos^2 t + \sin^2 t}{\cos^2 t} = \frac{a^2}{\cos^2 t}$

① $\int \left(\sqrt{a^2-x^2}\right) dx = \left| \begin{array}{l} x = a\sin t \\ dx = a\cos t\,dt \end{array} \right| =$

$= \int \sqrt{a^2\cos^2 t} \cdot a\cos t\,dt = a^2 \int \cos^2 t\,dt = a^2 \int \frac{1+\cos 2t}{2}\,dt$

$= \frac{a^2}{2}\left(\int dt + \int \cos 2t\,dt\right) = \frac{a^2}{2}\left(t + \frac{1}{2}\sin 2t\right) + C$

$= \frac{a^2}{2}\left(t + \frac{1}{2}\cdot 2\sin t\cos t\right) + C = \left| \begin{array}{l} \text{Iz smjene imamo:} \\ \sin t = \frac{x}{a} \end{array} \right| =$

$= \frac{a^2}{2}\left(\arcsin\frac{x}{a} + \frac{x}{a}\sqrt{1-\left(\frac{x}{a}\right)^2}\right) + C = \frac{a^2}{2}\cdot\arcsin\frac{x}{a} + \frac{ax}{2}\sqrt{\frac{a^2-x^2}{a^2}} + C$

$= \frac{a^2}{2}\cdot\arcsin\frac{x}{a} + \frac{x}{2}\sqrt{a^2-x^2} + C$

$\sin^2 t + \cos^2 t = 1$

$\sin^2 t = 1 - \cos^2 t$

② $\int \frac{dx}{(x^2-1)\sqrt{1-x^2}} = \left| \begin{array}{l} x = \sin t \\ dx = \cos t\,dt \end{array} \right| =$

$= \int \frac{\cos t\,dt}{(\sin^2 t - 1)\sqrt{1-\sin^2 t}} = \int \frac{\cos t\,dt}{(\sin^2 t - 1)\sqrt{\cos^2 t}} =$

$= \int \frac{dt}{\sin^2 t - 1} = -\int \frac{dt}{1-\sin^2 t} = -\int \frac{dt}{\cos^2 t} = -\operatorname{tg} t + C = -\frac{\sin t}{\cos t} + C$

$= -\frac{x}{\sqrt{1-x^2}} + C$

③ $\int \frac{dx}{\sqrt{(x^2+a^2)^3}} = \left| \begin{array}{l} x = a\cdot\operatorname{tg} t \\ dx = a\cdot\frac{1}{\cos^2 t}\,dt \end{array} \right| = \int \frac{\frac{a}{\cos^2 t}\,dt}{\sqrt{\left(\frac{a^2}{\cos^2 t}\right)^3}} =$

$= \int \frac{\frac{a}{\cos^2 t}}{\left(\frac{a}{\cos t}\right)^3} = \int \frac{\frac{a}{\cos^2 t}}{\frac{a^3}{\cos^3 t}} = \frac{a}{a^3}\int \cos t\,dt = \frac{1}{a^2}\sin t + C =$

$$= \left| tg\,t = \frac{x}{a} \right| =$$

$$\sin^2 t = \frac{\sin^2 t \quad : \cos^2 t}{\sin^2 t + \cos^2 t \quad : \cos^2 t} = \frac{tg^2 t}{tg^2 t + 1}$$

$$\sin t = \frac{tg\,t}{\sqrt{tg^2 t + 1}}$$

$$\underline{I} = \frac{1}{a^2} \cdot \frac{\frac{x}{a}}{\sqrt{\frac{x}{a} + 1}} + C = \frac{1}{a} \frac{\frac{x}{a}}{\frac{\sqrt{x^2+a^2}}{a}} + C = \frac{x}{a^2\sqrt{x^2+a^2}} + C$$

Za vježbu:

4. $\int x^7 \cdot \sqrt{1+x^2}\, dx$
5. $\int x^2 \sqrt{4-x^2}\, dx$
6. $\int \frac{dx}{x^3\sqrt{x^2+1}}$
7. $\int \frac{x^6}{\sqrt{x^2+1}}\, dx$

INTEGRALI HIPERBOLIČKIH FUNKCIJA

$$ch\,x = \frac{e^x + e^{-x}}{2} \qquad th\,x = \frac{sh\,x}{ch\,x}$$

$$sh\,x = \frac{e^x - e^{-x}}{2} \qquad cth\,x = \frac{ch\,x}{sh\,x}$$

$$(ch\,x)' = sh\,x \Rightarrow \int sh\,x\,dx = ch\,x + C$$

$$(sh\,x)' = ch\,x \Rightarrow \int ch\,x\,dx = sh\,x + C$$

$$(th\,x)' = \frac{1}{ch^2 x} \Rightarrow \int \frac{1}{ch^2 x} = th\,x + C$$

$(\mathrm{cth}\,x)' = \frac{1}{\mathrm{sh}^2 x} \Rightarrow \int \frac{1}{\mathrm{sh}^2 x} = \mathrm{cth}\,x + C$

$\boxed{\mathrm{ch}^2 x - \mathrm{sh}^2 x = 1}$

$\mathrm{ch}\,2x = \mathrm{ch}^2 x + \mathrm{sh}^2 x$

$\mathrm{sh}\,2x = 2\,\mathrm{sh}\,x \cdot \mathrm{ch}\,x$

$$\left.\begin{array}{l} \mathrm{ch}^2 x - \mathrm{sh}^2 x = 1 \\ \mathrm{ch}^2 x + \mathrm{sh}^2 x = \mathrm{ch}\,2x \end{array}\right\} +$$

$\mathrm{ch}^2 x = \frac{1 + \mathrm{ch}\,2x}{2}$ $\qquad$ $\mathrm{sh}^2 x = \frac{\mathrm{ch}\,2x - 1}{2}$

$\mathrm{th}\frac{x}{2} = t \Rightarrow \mathrm{sh}\,x = \frac{2t}{1-t^2}, \quad \mathrm{ch}\,x = \frac{1+t^2}{1-t^2}, \quad dx = \frac{2\,dt}{1-t^2}$

* ① $\int \mathrm{sh}\,x \cdot \mathrm{ch}\,5x\,dx = \int \frac{e^x - e^{-x}}{2} \cdot \frac{e^{5x} + e^{-5x}}{2} = \int \frac{e^{6x} + e^{-4x} - e^{4x} - e^{-6x}}{4}$

$= \frac{1}{4}\left(\int e^{6x}dx + \int e^{-4x}dx - \int e^{4x}dx - \int e^{-6x}dx\right.$

$= \frac{1}{4}\left(\int \frac{1}{6}e^{6x} - \frac{1}{4}e^{-4x} - \frac{1}{4}e^{4x} + \frac{1}{6}e^{-6x}\right) + C$

$= \frac{1}{24}e^{6x} - \frac{1}{16}e^{-4x} - \frac{1}{16}e^{4x} + \frac{1}{24}e^{-6x} + C$

$= \frac{1}{24}\left(e^{6x} + e^{-6x}\right) - \frac{1}{16}\left(e^{-4x} + e^{4x}\right) + C = \frac{1}{24}\,2\,\mathrm{ch}\,6x - \frac{1}{16}\cdot 2\,\mathrm{ch}\,4x + C$

$= \frac{1}{12}\,\mathrm{ch}\,6x - \frac{1}{8}\,\mathrm{ch}\,4x + C$

* ② $\int \mathrm{sh}^5 x \cdot \mathrm{ch}^{10} x\,dx = \int \mathrm{sh}\,x \cdot \mathrm{sh}^4 x \cdot \mathrm{ch}^{10} x\,dx =$

$= \int \mathrm{sh}\,x \cdot (\mathrm{sh}^2 x)^2 \cdot \mathrm{ch}^{10} x\,dx =$

$$\left| \begin{array}{l} ch^2x - sh^2x = 1 \\ \Rightarrow sh^2x = ch^2x - 1 \end{array} \right| = \int shx \cdot (ch^2x-1)^2 \cdot ch^{10}x\, dx = \left| \begin{array}{l} chx = t \\ shx\,dx = dt \end{array} \right| =$$

$$= \int (t^2-1)^2 \cdot t^{10}\, dt = \int (t^4 - 2t^2 + 1) \cdot t^{10}\, dt = \int (t^{14} - 2t^{12} + t^{10})\, dt =$$

$$= \frac{t^{15}}{15} - \frac{2t^{13}}{13} + \frac{t^{11}}{11} = \frac{ch^{15}x}{15} - \frac{2 \cdot ch^{13}x}{13} + \frac{ch^{11}x}{11} + C$$

③ $$\int \frac{dx}{shx} = \left| \begin{array}{l} th\frac{x}{2} = t \\ dx = \frac{2dt}{1-t^2} \end{array} \right| = \int \frac{\frac{2dt}{1-t^2}}{\frac{2t}{1-t^2}} =$$

$$= \int \frac{dt}{t} = \ln|t| + C = \ln\left|th\frac{x}{2}\right| + C$$

④ $$\int sh^4x\, dx = \int (sh^2x)^2 = \int \left(\frac{ch2x - 1}{2}\right) dx = \int \frac{ch^2 2x - 2ch2x + 1}{4} dx$$

$$= \frac{1}{4} \int (ch^2 2x - 2ch2x + 1)\, dx = \frac{1}{4}\left(\int ch^2 2x\, dx - 2\int ch2x\, dx + \int dx\right) =$$

$$= \frac{1}{4}\left(\int \frac{ch4x+1}{2} dx - 2 \cdot \frac{1}{2} sh2x + x\right) = \frac{1}{4}\left(\int \frac{ch4x}{2} dx + \int \frac{1}{2} dx - sh2x + x\right)$$

$$= \frac{1}{4} \cdot \left(\frac{1}{2} \cdot \frac{1}{4} sh4x + \frac{1}{2}x - sh2x + x\right) = \frac{1}{32} sh4x + \frac{3x}{8} - \frac{1}{4} sh2x + C$$

II način: $$\int sh^4x\, dx = \int \left(\frac{e^x - e^{-x}}{2}\right)^4 dx =$$

$$= \left| (a-b)^4 = a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4 \right|$$

za integrale oblika $\int f(\sqrt{a^2+x^2})\,dx$ uzeti smjenu: $x = a\,sh\,t$

za integrale oblika $\int f(\sqrt{a^2-x^2})\,dx$ uzeti smjenu: $x = a\,ch\,t$

⑤ $\int \sqrt{x^2-a^2}\,dx = \left| \begin{matrix} x = a\,ch\,t \\ dx = a\,sh\,t\,dt \end{matrix} \right| = \int \sqrt{a^2ch^2t - a^2}\cdot a\,sh\,t\,dt =$

$= \int \sqrt{a^2(ch^2t-1)}\cdot a\,sh\,t\,dt = a^2\int\sqrt{sh^2t}\cdot sh\,t\,dt = a^2\int sh^2t\,dt =$

$= a^2\int \frac{ch\,2t-1}{2}\,dt = \frac{a^2}{2}\left(\int ch\,2t\,dt - \int dt\right) = \frac{a^2}{2}\left(\frac{1}{2}sh\,2t - t\right) + C =$

$= \frac{a^2}{2}\left(\frac{1}{2}\cdot 2\,sh\,t\,ch\,t - t\right) + C = \left| \begin{matrix} ch\,t = \frac{x}{a} \\ sh\,t = \sqrt{ch^2t-1} \end{matrix} \right., \; t = Arch\frac{x}{a} = \ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2 - 1}\right) \Big|$

AREA FUNKCIJE:

$Ar\,sh\,x = \ln(x + \sqrt{x^2+1})$ ili | |

$Ar\,ch\,x = \ln(x + \sqrt{x^2-1})$

$\underline{I} = \frac{a^2}{2}\cdot\frac{x}{a}\sqrt{\left(\frac{x}{a}\right)^2-1} - \frac{a^2}{2}\cdot\ln\left(\frac{x}{a} + \sqrt{\left(\frac{x}{a}\right)^2-1}\right) + C =$

$= \frac{ax}{2}\cdot\frac{\sqrt{x^2-a^2}}{a} - \frac{a^2}{2}\ln\left(\frac{x}{a} + \sqrt{\frac{x^2}{a^2}-1}\right) + C$

za vježbu:

a) $\int ch^2x\,sh^2x\,dx$

b) $\int ch^3x\cdot sh^2x\,dx$

c) $\int \frac{sh\,x}{\sqrt{ch\,2x}}\,dx$

d) $\int \frac{2sh\,x + 3ch\,x}{4sh\,x + 5ch\,x}\,dx$

e) $\int\sqrt{x^2+a^2}\,dx$